A Model of Dynamic In-Consumption Social Interaction

Michelle Y. Lu[*], Ji Shen [†]

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Abstract

In digital content consumption, social interaction becomes a vital component that drives consumers' overall experience and willingness-to-pay for digital content. Despite the ubiquitous prevalence and growing trend of in-consumption social interaction, firms face new challenges: the loss of direct control over consumer experience and the uncertainty of real-time social interactions, potentially dampening consumer experience and hurting firms' profitability. Hence, managing social interaction becomes a critical problem for both firms and policymakers – who can participate, and how will social interaction change the paywall strategy? To answer these questions, we construct a continuous time model in which a consumer's willingness-to-pay for the product is dynamically shaped both by idiosyncratic shocks and social interactions with other consumers. We find that a firm benefits from allowing social interaction, even when downward social influence outweighs upward influence. The equilibrium price is lower, but demand is higher compared to the no-interaction benchmark. Moreover, when downward social influence is stronger than the upward one, the impact of social intensity on a firm's profitability hinges on a term we coin as "social elasticity"– the effect of a one-unit change in the intensity of social interactions on the reduction in demand. We show that a firm's profit can still increase with interaction intensity so long as the social elasticity is sufficiently small. We then extend the analysis to examine the interplay between a firm's innate content quality and social interactions and the case of purely interactive digital consumption.

Keywords: Social media; digital content; in-consumption social interactions; paywall strategy; dynamic continuous-time model

[*]China Europe International Business School, michellelu@ceibs.edu
[†]Peking University, jishen@gsm.pku.edu.cn

1 Introduction

Social interactions are essential to human life. With emerging technologies, social interactions can take place virtually and seamlessly intertwine with digital consumption, allowing people to mutually share real-time experiences, exchange ideas, and influence each other while consuming digital content. This type of in-consumption social interaction can take many forms. For example, viewers can share their real-time experiences via live comments on platforms such as YouTube's live streams, Twitch.tv, or via virtual forums on fan-based creator platform Patreon. On news sites such as The Wall Street Journal (WSJ) and The New York Times, readers can exchange ideas following news content. Online education sites like Coursera and interest-based clubs such as BookClub.com facilitate virtual communities for an enhanced learning experience through social influence.

In digital consumption, in-consumption social interactions become vital components that drive consumers' overall experience. It is reported that the number of interactive live comments on Bilibili exceeded 10 billion by 2021.[1] The number of people reading and leaving comments on WSJ increased more than 5%.[2] In the context of online news consumption, readers may enjoy more from the news content if they can encounter high-quality discussions and enjoy less if they read low-quality comments. Although firms like WSJ can prevent spams, the extent to which readers enjoy its content depends on the informativeness of articles as well as the comments provided by other readers so long as it enables social interactions.

Despite the ubiquitous prevalence and growing trend, the phenomenon of in-consumption social interactions imposes new challenges to firms: (1) Consumers' utility and willingness to pay for digital consumption depend not only on content quality but also on social influence among consumers, rendering firms to *lose direct control* over quality of experience; (2) A priori, it is *uncertain* how social interactions can shape every consumer's experience, as real-time social interactions can be highly random, possibly increasing or decreasing the consumer valuations.

Thus, it is not clear whether and how in-consumption social interactions can benefit firms. Firms adopt quite diverging strategies of dealing with in-consumption social interactions. In 2019, WSJ introduced a new comments strategy by limiting who can comment and on what articles, aiming to drive a higher quality of debate on its site. Amid media companies that allow virtual interactions,

[1] https://www.forbeschina.com/business/58500

[2] https://digiday.com/uk/the-wall-street-journal-tightens-comments-strategy-to-stimulate-higher-quality-debate/

others adopt an opposite approach. For example, since 2018, the Atlantic closed down comments due to uncertainty about how interactions shape site quality.[3] Netflix disallowed reviews and rating since 2018 despite consumers' need for live comments while streaming, which is enabled by a third-party Chrome extension.[4] In the same year, Youtube introduced paid channel membership for its creators to encourage social interactions in the fan community. Since 2023, Twitch also offers paid channel subscribers for exclusive chatroom.

Therefore, this research aims to fill in the knowledge gap by investigating the impact of in-consumption social interactions on firms' profitability. Specifically, we provide a cohesive framework to answer the following questions: (1) How will in-consumption social interactions affect user experience and willingness-to-pay, and thereby influencing the paywall strategy of digital content? (2) Can firms profit from social interactions even with the presence of negative social influence? How should firms change their paywall strategy compared with the case of no social interactions? (3) To what extent will social interactions influence firms' profitability?

To address these questions, we develop a dynamic continuous time model to capture the real-time nature of digital goods consumptions and a random matching model with stochastic type change to characterize in-consumption social interactions. The market consists of one monopoly firm that provides digital content and a continuum of consumers. In digital consumption, both idiosyncratic and social forces can change a consumer real-time experience. During a consumption session, consumers' instantaneous utilities are probabilistically drawn from a general continuous distribution, reflecting their own momentary needs and reactions to content variations, or probabilistically influenced by the utilities of other people who also consume the digital content. Social interactions can be either two-way (e.g., live chats, discussions) or one-way interactions (e.g. live comments) and can lead to *upward* social influence (a higher type consumer elevates the experience of a lower type consumer), *downward* social influence (a lower type consumer dampens the experience of a higher type consumer), *no* influence. Once consumption experience is realized, consumers decide whether to continue the experience or opt out for an outside option. Consumers maximize the discounted sum of instantaneous utilities. A monopoly firm sets the price to maximize the total profit. We

[3]https://www.theatlantic.com/letters/archive/2018/02/letters-comments-on-the-end-of-comments/552392/, https://www.theatlantic.com/technology/archive/2014/06/internet-comments-and-perceptions-of-quality/371862/

[4]https://www.forbes.com/sites/emmawoollacott/2018/07/06/why-netflix-wont-let-you-write-reviews-any-more/?sh=1eb13a726570

focus on a steady-state equilibrium in which price is time-invariant.

First, we show that social interactions shift consumer valuation distribution in terms of stochastic dominance, and adds an option value in a firm's pricing strategy. Two countervailing forces drive a firm's equilibrium price. If the firm sells to more consumers with lower willingness-to-pay, a marginal consumer with lower valuation drives down the price. However, the option value of social interactions decreases with consumers; willingness-to-pay. Compared with consumers of high valuation, consumers with lower willingness-to-pay value more on the option value of social interactions because they will be more likely to interact with high consumer types and gain more from interactions. The option value of social interactions drives up the price. In other words, the innate content quality and in-consumption social interactions are *substitutes* to each other. Combined the two effects, demand is less elastic than the no-interaction benchmark. That is, one unit increase in price change results in demand reduction smaller than one unit.

Second, we find that the firm can earn a higher profit by allowing social interactions, even when downward social influence is stronger than the upward one. The firm always finds better off with more consumers opting in. This result holds true even when downward social influence is stronger than the upward one. Moreover, the equilibrium price with social interactions can be lower due to the fact that the magnitude of price response is smaller than one. This result remains valid even when upward social influence is stronger than the downward influence, where the option value of social interactions can be much higher. This is because the positive effect of the option value on price is offset by the opposing force on the reduction in the marginal consumer's willingness-to-pay.

Last but not least, we explore how social interactions, both in terms of direction and intensity, impact a firm's profitability. Intuitively, as social interactions are more likely to positively influence consumers' valuations, the firm can charge a higher price and, consequently, earn a higher profit. This result is driven by the increased option value, as a stronger upward social influence rotates the post-interaction demand function upward. Regarding the intensity of social interactions (the relative weight of social interactions in driving consumer utility compared to idiosyncratic experience), when upward influence is stronger, a firm's profitability always increases with intensity. However, when downward influence is stronger, the firm's profitability depends on what we term as *"social elasticity"* – the effect of a one-unit change in the intensity of social interactions on the reduction in demand. Stronger downward social influence rotates the demand distribution clockwise, reducing demand

and negatively impacting the firm's profitability. In this scenario, the higher the intensity, the larger the amount by which demand reduces. Therefore, understanding the extent of demand loss in response to a unit increase in interaction intensity is crucial. We find that the firm's profit can still increase with interaction intensity as long as the social elasticity is sufficiently small."

We then extend the analysis to examine the interplay between a firm's innate content quality and social interactions and the case of purely interactive digital consumption.

Our research contributes to the literature on the economic and social impact of social interactions. Kuksov (2007) studies how social interactions influence the informativeness of brand use. The paper shows that brand can valuable even when social interactions convey no information. Manchanda et al. (2015) empirically analyze the economic effect of consumer social interactions in online communities and confirm that consumers who interact more contribute higher revenue to the firm. Zhang et al. (2020) investigates the design of social interactions on online learning platform and assess the consequential influence of such interactions on education quality. They show that promoting social interaction among students has a positive impact on learning outcomes. Zhang et al. (2020) proposes a measure, moment-to-moment synchronicity (MTMS), to capture the synchronicity between viewers' in-consumption engagement and movie content. Their research demonstrates that MTMS significantly predicts viewers' post-consumption evaluation of movies. We contribute to the understanding of this phenomenon by providing a theoretical framework that allows to analyze different elements of social interaction on equilibrium outcomes.

We also provide new insights to the literature on digital content and paywall strategy. Amaldoss et al. (2021) and Wang et al. (2024) study a digital content provider's paywall strategy when it balances between ad and content revenue. Lambrecht and Misra (2017), Pattabhiramaiah et al. (2019) and Chae et al. (2023) empirically investigate the impact of digital paywall on a publisher's profit streams. Chiou and Tucker (2013) studies the effect of paywall strategy on readership reduction.

by examining a new phenomenon of in-consumption social interactions.

The rest of the paper is organized as follows: We introduce the model in Section 2. In the main analysis, Section 3 examines the key insights of the strategic impact of in-consumption social interactions and Section 4 investigates how social interactions influence a firm's profitability. Then we extend the analysis in Section 5. We conclude and acknowledge our limitations in Section 6.

2 Model

We construct a dynamic continuous time model to characterize digital consumption and in-consumption social interactions. At each time t, a monopoly firm sells digital content at price p_t to a unit mass of consumers, each index by $i \in [0,1]$. At each time t, a consumer either consumes the digital content or takes the outside option u_0.

2.1 Real-time Digital Consumption

Let θ_{it} be consumer i's experience and thereby willingness-to-pay for digital content at time t. The state variable θ_{it} follows a well-defined distribution $F(\cdot)$ on a support $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$. A standard technical assumption on log-concavity of $1 - F(\cdot)$ applies throughout the paper.

Two forces can change a consumer i's real-time experience θ_{it}. One is an idiosyncratic force, e.g., momentary leisure or informational need for consuming content, consumer reactions to temporal variations in the content dynamics (Zhang et al., 2020). Moreover, with emerging technologies that enable in-consumption social interactions, consumers' experience can also change depending on whom they encounter or interact with, enhancing or dampening their enjoyment as consumption proceeds (Ramanathan and McGill, 2007, Zhang et al., 2020, Liu and Kwon, 2022). We consider in-consumption social interaction as a second social force.

Idiosyncratic Experience

The first force comes from consumer idiosyncratic experience. During a short period $[t, t+dt)$, consumer type may change with probability κdt such that a new value is drawn according to distribution function $F(\cdot)$ on $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$.

With probability $1 - \kappa dt$, a consumer does not experience an idiosyncratic shock and sticks to the last-period experience. That is, we consider the following transition probabilities:

$$\begin{aligned} \Pr\left(\theta_{t+dt} = \xi | \theta_t = \xi\right) &= 1 - \kappa dt, \\ \Pr\left(\theta_{t+dt} = \xi' | \theta_t = \xi\right) &= \kappa f\left(\xi'\right) dt. \end{aligned} \tag{1}$$

Here, $\kappa \in (0, \infty)$ can be interpreted as intensity of idiosyncratic shocks, depending on the nature of digital content or consumption habits. For example, a movie with a lot of surprise and suspense leads to a higher κ.

In-consumption Social Interactions

The second source of experience change is induced by social interactions. In digital content consumption at time t, a consumer with contemporaneous experience $x \in \Theta$ randomly encounters another consumer with $y \in \Theta$ with probability $\lambda \cdot m_t(y) \cdot dt$, where λ represents the intensity of social interactions and $m_t(y)$ denotes the *equilibrium* type distribution. Both exogenous and endogenous reasons influence an occurrence of social interactions. For example, bandwidth limits may constrain the display of all live comments or a stringent scrutiny technology is adopted to moderate online conversations. These two factors can reduce the level of λ. Furthermore, whether a consumer can encounter a particular type of consumer y is proportional to its density $m_t(y)$. This type distribution should take into account consumers' equilibrium behavior.

We follow the Psychology literature about "coherence" effect in social interactions such that one's experience move up and down together with others (Hatfield et al., 1993, Neumann and Strack, 2000, Ramanathan and McGill, 2007). When a consumer with x encounters a consumer with y, assume with no loss of generality that $x > y$. In an infinitely small period dt, one of the three possible outcomes happens once an interaction occurs: the low-type consumer y's experience is enhanced to x (*upward* social influence), x's experience is dampened to y (*downward* social influence), no social influence takes place. These events happen with probabilities ω_U, ω_D, and $1-\omega_U-\omega_D$, respectively. The probabilities also capture variations in the social aspects of digital content: relative strengths of upward and downward social influences may vary across different types of content. For example, in news consumption, consumers are more likely to be influenced when reading experience-dampening comments; while in entertainment consumption, consumers are more likely to immune from them, and experience-enhancing comments are more likely to take effect.

We would like to make three notes. First, the continuous-time framework makes the model more tractable compared with its discrete-time counterpart because the consumer type is changed either by an idiosyncratic shock or by social interactions in an infinitely small time period.

Viewed in this way, a second advantage of adopting the stochastic continuous-time modeling approach lies in that it flexibly accommodates *both two-way and one-way* interactions. In two-way communications (e.g., discussions, forums, communities, chats, etc.), the two counterparties can mutually influence each other's experience. In one-way interaction (e.g., comments without response), one party unilaterally influences the counterparty. In this case, one can interpret ω_U (ω_D) as the probability that a high-type (low-type) consumer happens to be an influencer and succeeds in influencing the counterpart and $1 - \omega_U - \omega_D$ as the probability that the receiver is not influenced.

Third, our model can be extended to incorporate additional psychological values that a consumer may obtain from successfully influencing others or may subtract when meeting a lower-type consumer. Adding these values would strengthen our results. To reflect our core idea, we omit them for the simplicity of the model.

2.2 Consumer Decision and Population Distribution

During a short period $[t, t + dt)$, a consumer either opts in and consumes the digital content ($\chi_{it} = 1$) or opts out and takes the outside option ($\chi_{it} = 0$). Then a consumer's instantaneous utility at time t is given by,

$$(\theta_{it} - p_t)\,\mathbf{1}(\chi_{it} = 1) + u_0 \cdot \mathbf{1}(\chi_{it} = 0)\,. \tag{2}$$

If a consumer opts in, the content value θ_{it} is realized, and the consumer i enjoys a net payoff $\theta_{it} - p_t$. If a consumer opts out, the consumer enjoys an outside option with a value of $u_0 \in \left[\underline{\theta}, \overline{\theta}\right]$.

Once consumers are non-users of the platform, they cannot view content nor participate in social interactions. Their valuations for content are only driven by idiosyncratic shocks as described in equation (1). Then they decide whether to continue staying outside or to opt in for content. Without loss of generality, we assume that consumers choose to be users when they are indifferent between consuming content and opting out for the outside option.

Let Θ_{0t} and Θ_{1t} be the set of consumers who choose to be non-buyers or buyers, respectively.

$$\begin{aligned}
\Theta_{0t} &= \left\{\theta_i \in \left[\underline{\theta}, \overline{\theta}\right] | x_{it} = 0, i \in [0,1]\right\}, \\
\Theta_{1t} &= \left\{\theta_{it} \in \left[\underline{\theta}, \overline{\theta}\right] | x_{it} = 1, i \in [0,1]\right\},
\end{aligned}$$

so that the two subsets are disjoint and the union of them is exactly the whole type space:

$$\Theta_{0t} \cap \Theta_{1t} = \varnothing, \Theta_{0t} \cup \Theta_{1t} = \left[\underline{\theta}, \overline{\theta}\right].$$

Let the type population density function for all consumers be $m_t(\cdot)$. That is, the measure of consumers with types in interval $(\theta, \theta + d\theta)$ amounts to $m_t(\theta)\, d\theta$. Since consumers of all types should be summed up to the total population, we have

$$1 = \int_{\Theta_{1t}} m(\theta)\, d\theta.$$

Accordingly, we can have $M_t(\theta)$ the measure of users with types no more than θ.

2.3 Firm's Pricing Strategy

The monopoly firm sets price trajectory $\{p_t\}_{t=0}^{\infty}$ to maximize the total profit:

$$\max_{p_t} \Pi = p_t \cdot \sharp \{i \in [0,1] \,| \chi_{it} = 1\}.$$

2.4 Equilibrium Definition

We focus on the steady-state equilibrium, which should be comprised of three parts:

- Given the population distribution of consumers and the content price, each individual consumer decides whether or not to consume the content in order to maximize the lifetime discounted sum of flow utility;
- By aggregating each consumer's optimal decision, we obtain the population distribution.
- Given the population distribution and each consumer's optimal decision, we obtain the profit-maximizing price.

3 Equilibrium Analysis

At each time t, a consumer either consumes the content or takes the outside option. A consumer's objective is to maximize the following discounted sum of instantaneous utility:

$$\mathbb{E}\left[\int_t^\infty e^{-r(\tau-t)}\left[(\theta_{i\tau}-p)\,\mathbf{1}\left(\chi_{i\tau}=1\right)+u_0\mathbf{1}\left(\chi_{i\tau}=0\right)\right]d\tau\right], \tag{3}$$

We focus on the steady-state equilibrium, where the type distribution of consumers stays constant over time. Given the steady-state population distribution of consumers and the price, we derive their optimal strategy. Denote V_1 and V_0 as the expected discounted sum of instantaneous utility if a consumer chooses to become a user and a non-user, respectively. The optimal strategy comes from the comparison between the expected discounted sum of instantaneous utilities of opting in and out of the platform, which is given by the following rule,

$$\chi=\begin{cases}1,\text{ if } V_1\geq V_0\\ 0,\text{ if } V_1\ <V_0\end{cases} \tag{4}$$

Thus, the value function is given by

$$V=\max\left\{V_1\ ,V_0\right\}. \tag{5}$$

3.1 Benchmark: No Social Interaction

In this subsection, we first study a benchmark case without social interactions. Consumers wait for exogenous type switching shock, which occurs to an individual at Poisson rate κ,

We further specify V_0. A non-user receives an instantaneous utility flow at rate u_0, and waits for an exogenous shock to get the valuation changed:

$$V_0=\beta dt\cdot u_0+(1-\beta dt)\left[\kappa dt\int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}}V\left(\theta'\right)dF\left(\theta'\right)+(1-\kappa dt)\,V_0\ \right], \tag{6}$$

where β is the discount factor.

In the steady state, we have $\frac{d}{dt}V_0 = 0$, which can be further rearranged as follows,

$$V_0 = \frac{\beta}{\beta+\kappa} \cdot u_0 + \frac{\kappa}{\beta+\kappa} \cdot \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V\left(\theta'\right) dF\left(\theta'\right) \equiv \widehat{V_0}. \tag{7}$$

Equation (7) shows that a non-user's value function is a weighted average of the two components: the flow payment from outside option weighted by $\frac{\beta}{\beta+\kappa}$ and the option value induced by an exogenous shock weighted by $\frac{\kappa}{\beta+\kappa}$. Moreover, $V_0(\theta)$ is independent of θ.

If the consumer opts in, the value function becomes

$$\begin{aligned} V_1(\theta) &= \beta dt \cdot (\theta - p) + (1-\beta dt)\left[\kappa dt \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V\left(\theta'\right) dF\left(\theta'\right) + (1-\kappa dt) V_1(\theta)\right] \\ &= \frac{\beta(\theta-p)}{\beta+\kappa} + \frac{\kappa}{\beta+\kappa} \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V\left(\theta'\right) dF\left(\theta'\right). \end{aligned} \tag{8}$$

Taking difference, we find

$$V_1(\theta) - V_0 = \frac{\beta}{\beta+\kappa}(\theta - p - u_0) \begin{cases} \geq & 0 \text{ if } \theta \geq (p+u_0) \\ < & \qquad\quad < \end{cases}.$$

Hence,

$$V(\theta) = \max\{V_1(\theta), V_0(\theta)\} = \begin{cases} V_1(\theta) & \text{if } \theta \geq (p+u_0) \\ V_0 & \qquad\ < \end{cases}.$$

The consumer's optimal choice is myopia in the sense that one simply compares the instantaneous payoff from participating (which gives him $\theta - p$) and not (which yields u_0).

We characterize the equilibrium in the first lemma and relegate all proofs in the Appendix.

Lemma 1. *Without social interactions, the optimal price p^*_{NI} is determined by*

$$p^*_{NI} = \frac{1 - F\left(p^*_{NI} + u_0\right)}{f\left(p^*_{NI} + u_0\right)}. \tag{9}$$

Consumers with $\theta > \theta^*_{NI}$ opt in to consume digital content, where $\theta^*_{NI} = p^*_{NI} + u_0$, where subscript "NI" stands for the case of no social interactions.

The no-interaction equilibrium corresponds to the classic monopoly pricing problem. Consumers' decision rule remains the same as in the myopic one-period game.

3.2 Social Interaction Equilibrium

In the proceeding section, let us examine how the introduction of social interactions impact consumers' value functions. During a short period $[t, t+dt)$, a consumer's the sum of expected discounted payoffs can be broken down to the present value and the future value given that the type is changed by an exogenous shock, changed by social interaction, or unchanged after social interaction.

$$\begin{aligned} V_1(\theta) = \; & \underbrace{\beta dt \cdot (\theta - p)}_{\text{instantaneous consumption}} + (1-\beta dt) \left[\underbrace{\kappa dt \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V(\theta')\, dF(\theta')}_{\text{idiosyncratic shock}} \right. \\ & \underbrace{+\lambda dt \int_{y \in \Theta_1, y \leq \theta} m_1(y) \left[\omega_D V(y) + \omega_U V_1(\theta) + (1-\omega_U-\omega_D) V_1(\theta)\right] dy}_{\text{social interaction with a lower type}} \\ & \underbrace{+\lambda dt \int_{x \in \Theta_1, x \geq \theta} m_1(x) \left[\omega_U V(x) + \omega_D V_1(\theta) + (1-\omega_D-\omega_U) V_1(\theta)\right] dx}_{\text{social interaction with a higher type}} \\ & \left. \underbrace{+\left(1-\kappa dt - \lambda \widehat{M}_1 dt\right) V_1(\theta)}_{\text{type unchanged}} \right], \end{aligned} \tag{10}$$

where $\Theta_1 \equiv \{\theta : \chi(\theta) = 1\}$ denotes a set of paid consumers. For completeness, let us also define $\Theta_0 \equiv \{\theta : \chi(\theta) = 0\}$ as a set of consumers who opt out.

The first term in equation (10) captures the instantaneous consumption flow in the current period. The second term (terms included in the bracket) captures what takes place in the next short period of time dt. Firstly, a consumer receives a type-switching shock. The second and third line denote the change in a consumer's value function due to social interactions. The second line denotes the case where the consumer encounters another one with a lower type, say y. Then the consumer type is driven down by the counterparty with probability ω_D and ends up with a value function $V(y)$, or the counterparty's type is elevated up by the consumer with probability ω_U, or no one changes the other's type and they remain their type unchanged with probability $1-\omega_D-\omega_U$. In the last two cases, the consumer value function is still $V_1(\theta)$. The third line, on the other hand, denotes the case where a consumer encounters another one with a higher type, say x. Similarly, there are three subcases then: either the consumer is affected by the counterparty so that the new type becomes x with probability ω_U and ends up with a new value function $V(x)$, , the consumer drives down the counterparty with probability ω_D, or no one changes anyone with the remaining probability. In the last two cases, the consumer's value function remains unchanged. If all the events described above have not happened, the consumer ends up with the same type as the last minute and his value function is still $V_1(\theta)$. This is reflected in the last line.

Without social interaction, population distribution is simply the initial one $F(\cdot)$. Notice that social interactions change the population distribution where consumer valuation is drawn, $M_1(\cdot)$, where $M_1(\theta)$ is the cumulative population size of users up to type θ with associated density function $m_1(\cdot)$. We also denote the total measure of users as $\widehat{M}_1$. More specifically,

$$M_1(\theta) = \int_{y\in\Theta_1, y\le\theta} m_1(y)\,dy, \tag{11}$$

$$\widehat{M}_1 = \int_{\theta\in\Theta_1} m_1(\theta)\,d\theta. \tag{12}$$

Then the population size of users whose type are above θ is given by

$$\widehat{M}_1 - M_1(\theta) = \int_{x\in\Theta_1, x\ge\theta} m_1(x)\,dx. \tag{13}$$

Then we rearrange $V_1(\theta)$ as follows,

$$\begin{aligned}\left(\beta+\kappa+\lambda\widehat{M}_1\right)V_1(\theta) &= \beta(\theta-p)+\kappa\int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V(\theta')\,dF(\theta') \\ &\quad +\lambda\omega_D\int_{y\in\Theta_1,y\le\theta} m_1(y)V(y)\,dy+\lambda(1-\omega_D)V_1(\theta)M_1(\theta) \\ &\quad +\lambda\omega_U\int_{x\in\Theta_1,x\ge\theta} m_1(x)V(x)\,dx+\lambda(1-\omega_U)V_1(\theta)\left[\widehat{M}_1-M_1(\theta)\right]\end{aligned} \tag{14}$$

Taking total differentiation w.r.t. θ and rearranging the result, we can have

$$\frac{dV_1(\theta)}{d\theta} = \frac{\beta}{\beta+\kappa+\lambda\omega_U\widehat{M}_1-\lambda\Delta M_1(\theta)} > 0. \tag{15}$$

Since V_0 is independent of θ and $V_1(\theta)$ increases in θ, we know that the optimal consumer decision can be characterized by a threshold strategy: There must exist a cutoff point, denoted by

θ_1^*, such that

$$V_1(\theta)\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} V_0 \text{ whenever } \theta \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} \theta_1^*, \tag{16}$$

It follows that

$$M_1(\theta) = \int_{\theta_1^*}^{\theta} m_1(y)\,dy, \theta \in \left[\theta_1^*, \overline{\theta}\right]. \tag{17}$$

We construct the expressions for the value functions in the Appendix and present the formulas as follows,

$$\begin{aligned} V_0(\theta) &= V_0 = u_0 + \int_{\theta_1^*}^{\overline{\theta}} \frac{\kappa\left[1 - F(x)\right]}{\beta + \kappa + \lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx, \forall\theta \in \left[\underline{\theta}, \overline{\theta}\right], \\ V_1(\theta) &= V_0 + \int_{\theta_1^*}^{\theta} \frac{\beta}{\beta + \kappa + \lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx, \theta \in \left[\theta_1^*, \overline{\theta}\right]. \end{aligned}$$

Then we can obtain consumers' equilibrium decision in the next lemma.

Lemma 2. *Consumers opt in to consume digital content when $\theta \geq \theta_1^*$ and opt out when $\theta < \theta_1^*$, where θ^* is implicitly determined by*

$$p + u_0 = \theta_1^* + \underbrace{\int_{\theta_1^*}^{\overline{\theta}} \frac{\lambda\omega_U\left[\widehat{M}_1 - M_1(x)\right]}{\beta + \kappa + \lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx}_{\text{Option Value of Social Interactions}}, \tag{18}$$

where $\Delta \equiv \omega_U - \omega_D$.

Let us provide some intuitions for equation (18). When the firm increases its price, the type of the marginal customer should be increased as high price drives low type consumers out of the market. If customers follow the myopic rule, as they did in the benchmark case with no social interaction, then one unit of price increase leads to a unit of decrease in the marginal customer's type. However, compared with the benchmark case of no interaction, the presence of social interaction leads to an option value of opting in. Note that this option value is evaluated at the marginal consumer's willingness-to-pay. To understand how a price change will affect marginal consumer's decision, let us define the option value of social interactions as follows,

$$I\left(\theta_1^*\right) \equiv \int_{\theta_1^*}^{\overline{\theta}} \frac{\lambda\omega_U\left[\widehat{M}_1 - M_1\left(x\right)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1\left(x\right)}dx \tag{19}$$

To understand the marginal consumer's decision, we take the derivative of $I\left(\theta_1^*\right)$ with respect to θ_1^*:

$$\frac{dI(\theta_1^*)}{d\theta_1^*} = -\frac{1}{1+\frac{\beta+\kappa}{\lambda\omega^U\widehat{M}_1}} \in (-1,0) \tag{20}$$

This result shows that when price is increased by ΔP, the marginal type is increased by an amount smaller than one unit. In other words, demand is *less elastic* compared with the no-interaction benchmark.

Next, knowing equilibrium consumer decision, we solve for the equilibrium population distribution. We use $m_1\left(\theta\right)$ to denote the density of users, that is, users' population size in the region $\left(\theta, \theta+d\theta\right)$ is given by $m_1\left(\theta\right)d\theta$. Similarly, we use $m_0\left(\theta\right)$ to denote the density of non-users.

Since one must be either a user or a non-user, we should have the following identity

$$\int_{\theta=\underline{\theta}}^{\overline{\theta}} \left[m_0\left(\theta\right)+m_1\left(\theta\right)\right]d\theta = 1. \tag{21}$$

We first derive the expression for $m_0\left(\theta\right)$, which takes non-degenerate form on interval $[\underline{\theta}, \theta_1^*]$. Let's focus on non-users with types ranging in interval $\left(\theta, \theta+d\theta\right)$ at time t, amounting to $m_0\left(\theta\right)d\theta$ in total. Consider a short time interval during $\left(t, t+dt\right)$.

- (i) A fraction κdt of customer (either user or non-user) experience a shock in their types, so they flow out from this group (with total measure $\kappa dt \cdot m_0\left(\theta\right)d\theta$).

- (ii) Some customers (either user or non-user) experience a shock in their types and their new types happen to lie in this interval, so these customers flow into this group (with total measure $\kappa dt \cdot f\left(\theta\right)d\theta$).

In the steady-state equilibrium, inflows must be equal to outflows so that

$$\kappa dt \cdot m_0\left(\theta\right)d\theta = \kappa dt \cdot f\left(\theta\right)d\theta \Rightarrow m_0\left(\theta\right) = f\left(\theta\right), \forall\theta \in \left[\underline{\theta}, \theta_1^*\right]. \tag{22}$$

We can therefore obtain the measure of non-users:

$$\widehat{M}_0 = \int_{\underline{\theta}}^{\theta_1^*} m_0(\theta)\, d\theta = \int_{\underline{\theta}}^{\theta_1^*} f(\theta)\, d\theta = F(\theta_1^*) \Rightarrow \widehat{M}_0 = F(\theta_1^*) \,. \tag{23}$$

We now derive the expression for $m_1(\theta)$, which takes non-degenerate form on interval $\left[\theta_1^*, \overline{\theta}\right]$. Still focus on users with types in interval $(\theta, \theta + d\theta)$, whose total measure amounts to $m_1(\theta)\, d\theta$. Consider a short time interval during $(t, t + dt)$.

- (i) A fraction κdt of users experience a shock in their types (with total measure $\kappa dt \cdot m_1(\theta)$).
- (ii) Such a user meets another user with type $y < \theta$ with prob. $\lambda m_1(y)\, dt$. After meeting, his type becomes y with prob. ω_D and thus flow out from this interval, or his counterparty's type becomes θ with prob. ω_U and thus a new individual flows into this interval, or both remains their type with prob. $(1 - \omega_U - \omega_D)$ and no change in this situation.
- (iii) Such a user meets another user with type $x > \theta$ with prob. $\lambda m_1(x, t)\, dt$. After meeting, his type becomes x with prob. ω_U and thus flow out from this interval, or his counterparty's type becomes θ with prob. ω_D and thus a new individual flows into this interval, or both remains their type with prob. $(1 - \omega_U - \omega_D)$.

Hence,

$$\begin{aligned} m_1(\theta)\, d\theta &= (1 - \kappa dt)\, m_1(\theta)\, d\theta + \kappa dt \cdot f(\theta)\, d\theta + \lambda dt \cdot m_1(\theta)\, d\theta\, (-\omega_D + \omega_U) \int_{y \in \Theta_1, y \leq \theta} m_1(y)\, dy \\ &\quad + \lambda dt \cdot m_1(\theta)\, d\theta\, (-\omega_U + \omega_D) \int_{x \in \Theta_1, x \geq \theta} m_1(x)\, dx. \end{aligned}$$

which can be simplified to the following equation:

$$\kappa m_1(\theta) = \kappa f(\theta) + \lambda \Delta m_1(\theta)\, M_1(\theta) - \lambda \Delta m_1(\theta) \left[\widehat{M}_1 - M_1(\theta)\right]. \tag{24}$$

We can solve for the type distribution of users as follows

$$M_1(\theta) = \begin{cases} \frac{1}{2}\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right) - \frac{1}{2}\sqrt{\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}\left[F(\theta) - F(\theta_1^*)\right]}, \text{ if } \Delta > 0 \\ F(\theta) - F(\theta_1^*), \text{ if } \Delta = 0 \\ \frac{1}{2}\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\sqrt{\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}\left[F(\theta) - F(\theta_1^*)\right]}, \text{ if } \Delta < 0 \end{cases}, \forall \theta \in \left[\theta_1^*, \overline{\theta}\right]. \tag{25}$$

Obviously, $M_1(\theta)$ is strictly increasing in θ.

To obtain $\widehat{M}_1$ (the total measure of users), the following equation should hold:

$$\begin{aligned} 0 &= -\left(\lambda\Delta\widehat{M}_1 + \kappa\right) M_1\left(\overline{\theta}\right) + \kappa\left[1 - F(\theta_1^*)\right] + \lambda\Delta\left[M_1\left(\overline{\theta}\right)\right]^2 \\ &= -\left(\lambda\Delta\widehat{M}_1 + \kappa\right)\widehat{M}_1 + \kappa\left[1 - F(\theta_1^*)\right] + \lambda\Delta\left(\widehat{M}_1\right)^2 = -\kappa\widehat{M}_1 + \kappa\left[1 - F(\theta_1^*)\right] \\ &\Rightarrow \widehat{M}_1 = 1 - F(\theta_1^*). \end{aligned} \tag{26}$$

The last equation, together with (23), confirm the identity (21).

The density of users is given by

$$m_1(\theta) = \frac{f(\theta)}{\sqrt{\left(\lambda\Delta\widehat{M}_1 + \kappa\right)^2 - 4\kappa\lambda\Delta\left[F(\theta) - F(\theta_1^*)\right]}}. \tag{27}$$

After we obtain equilibrium population distributions, characterized in equations (25)-(27), we can then examine how social influence shapes population distribution.

Lemma 3. *Conditional on Θ_1, the equilibrium population distribution with social interaction $M_1(\theta)$ and the truncated population distribution without social interaction $F(\theta)$ can be ranked in terms of stochastic dominance:*

$$\frac{M_1(\theta)}{\widehat{M}_1} \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \frac{F(\theta) - F(\theta_1^*)}{1 - F(\theta_1^*)}, \textit{ if } \Delta \left\{ \begin{matrix} > \\ = \\ < \end{matrix} \right\} 0 \quad \begin{matrix} \textit{stronger upward influence} \\ \textit{neutral influence} \\ \textit{stronger downward influence} \end{matrix} \quad . \tag{28}$$

The probability density functions $m_1(\theta)$ and $f(\theta)$ can be ranked in terms of monotone likelihood *ratio:*

$$\frac{d}{d\theta}\frac{m_1(\theta)}{f(\theta)}\left\{\begin{array}{c}>\\=\\<\end{array}\right\}0,\ \mathit{if}\ \Delta\left\{\begin{array}{c}>\\=\\<\end{array}\right\}0. \tag{29}$$

When $\Delta > 0$ such that the upward social influence is stronger than the downward influence ($\omega_U > \omega_D$), $\frac{m_1(\theta)}{f(\theta)}$ increases over the interval $\left[\theta_1^*, \overline{\theta}\right]$. When $\Delta < 0$ such that the downward social influence more likely to take place ($\omega_D > \omega_U$), $\frac{m_1(\theta)}{f(\theta)}$ decreases over the interval $\left[\theta_1^*, \overline{\theta}\right]$.

Moreover, it is well known that MLRP leads to the first-order stochastic dominance. Hence, when $\Delta > 0$, $\frac{M_1(\theta)}{\widehat{M}_1}$ first-order stochastically dominates $\frac{F(\theta)-F\left(\theta_1^*\right)}{1-F\left(\theta_1^*\right)}$: $1-\frac{M_1(\theta)}{\widehat{M}_1} > 1-\frac{F(\theta)-F\left(\theta_1^*\right)}{1-F\left(\theta_1^*\right)}$. Upward communication allocates more types to the upper tail relative to the underlying distribution $F(\theta)$. When $\Delta < 0$, the reverse holds: $\frac{M_1(\theta)}{\widehat{M}_1}$ is first-order stochastically dominated by $\frac{F(\theta)-F\left(\theta_1^*\right)}{1-F\left(\theta_1^*\right)}$.

Given the equilibrium consumer behavior and the subsequent population distribution, we then solve for the firm's profit-maximizing problem:

$$\max_p \Pi = p \cdot \widehat{M}_1 = p\left(1 - F\left(\theta_1^*\right)\right) \tag{30}$$

It is worth noting that the demand $\widehat{M}_1$ does not depend on $M_1(\theta)$ but the cutoff θ_1^*. The impact of the new population distribution with social influence enters through the participation cutoff θ_1^*, which is given by equation (18) in Lemma 2. Recall that equation, equivalently, we can rewrite the firm's profit-maximizing problem in terms of θ_1^* as follows,

$$\max_{\theta_1^*} \Pi = \left(\theta_1^* - u_0 + I(\theta_1^*)\right)\left(1 - F\left(\theta_1^*\right)\right) \tag{31}$$

We characterize the social interaction equilibrium in the following proposition.

Proposition 1. *The steady-state equilibrium consists of the cutoff θ_1^*, the distributions $m_1(\theta)$ and $m_0(\theta)$, and the price p^*, which should satisfy:*

- $\frac{1+\frac{dI(\theta_1^*)}{d\theta_1^*}}{\theta_1^*-u_0+I(\theta_1^*)} = \frac{1-F(\theta_1^*)}{f(\theta_1^*)}$, *where* $I(\theta_1^*) = \int_{\theta_1^*}^{\overline{\theta}} \frac{\lambda\omega_U\left[\widehat{M}_1-M_1(x)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1-\lambda\Delta M_1(x)}dx$;
- $p^* = \theta_1^* - u_0 + I(\theta_1^*)$;

- $m_1(\theta) = \frac{f(\theta)}{\sqrt{\left(\lambda\Delta\widehat{M}_1+\kappa\right)^2 - 4\kappa\lambda\Delta\left[F(\theta)-F\left(\theta_1^*\right)\right]}}$, $m_0(\theta) = f(\theta)$, *and* $\int_{\theta=\underline{\theta}}^{\overline{\theta}} \left[m_0(\theta) + m_1(\theta)\right] d\theta = 1.$

Equilibrium Pricing

Different from classic monopoly pricing problem, there are two countervailing forces drive the optimal pricing. To see this, we have

$$\frac{dp^*(\theta_1^*)}{d\theta_1^*} = 1 + \frac{dI(\theta_1^*)}{d\theta_1^*} = \frac{1}{1 + \frac{\lambda\omega_U}{\beta+\kappa}\widehat{M}_1} \in (0,1) \tag{32}$$

Similar to the classic monopoly pricing without social interaction, as θ_1^* decreases, consumers' valuation for the content decreases, resulting in a lower price. Nonetheless, with social interaction, as θ_1^* decreases, the option value $I(\theta_1^*)$ increases. That is, as consumers' valuation for the product becomes lower, they expect that they will gain more from interactions with more high-type consumers. In a sense, product and social interaction are *substitutes* to each other. Combined the two above countervailing forces together, the price best response $p(\theta_1^*)$ to marginal change in demand is positive but smaller than one.

Compare with the no-interaction benchmark

To understand the impact of social interactions on the equilibrium outcomes, let us compare the social interaction equilibrium with the no interaction benchmark. We first describe the result below.

Proposition 2. *Compared with the no-interaction benchmark, social interaction always results in more consumers* $\theta_1^* < \theta_{NI}$*, a lower price* $p^* < p_{NI}$*, and a higher profit* $\Pi^* > \Pi_{NI}$

To gain an intuition, let us rearrange the first-order condition of equation (31) such that θ_1^* is determined by:

$$\left[\underbrace{1 - F(\theta_1^*) - (\theta_1^* - u_0)f(\theta_1^*)}_{\text{no-interaction benchmark}}\right] + \left[\underbrace{\frac{dI(\theta_1^*)}{d\theta_1^*}}_{<0}(1 - F(\theta_1^*)) - f(\theta_1^*)I(\theta_1^*)\right] = 0 \tag{33}$$

The first bracket represents the condition of θ_{NI}. As shown earlier, $\frac{dI(\theta_1^*)}{d\theta_1^*} < 0$, we must have $\theta_1^* < \theta_{NI}$. That is, the platform always finds better off to welcome more consumers to opt in. And this result holds true even when downward social influence is stronger than the upward influence ($\Delta < 0$). This is because the marginal consumer θ_1^* is the lowest type in social interactions, which will not be driven even lower.

Moreover, the equilibrium price with social interaction should be lower due to the fact that the magnitude of price response is smaller than one. This result holds true even when upward social influence is stronger than the downward influence ($\Delta > 0$) where the option value of social interactions can be much higher. This is because the positive effect on price is offset by the countervailing force on the reduction in θ_1^*.

Combined the demand and price effect together, we find that the platform can earn a higher profit by allowing for social interactions even when downward social influence is stronger than the upward one. This result is counterintuitive. One might think when negative social influence drives the user experience, the firm is better off disallowing social interactions, which is consistent with practices of the Atlantic and Netflix. Nonetheless, this layman intuition holds true when the firm allows social interactions while fixing the price at the level of no social interactions.

4 Comparative Statics

In this section, we will examine how social interactions will impact the equilibrium profit: *direction* and *intensity* of social interactions.

4.1 Direction of Social Influence

As shown in Lemma 3, the direction of social interactions Δ qualitatively shapes distribution of post-interaction consumer valuations. In the next result, we examine how the direction of social influence will impact the equilibrium profit and pricing strategy.

Proposition 3. *The equilibrium profit and price increase in* Δ.

Intuitively, as social interactions are more likely to positively influence consumers' valuations, the firm can charge a higher price and earn a higher profit. The mechanism that drives the result is through the increased option value as a stronger upward social influence rotates the post-interaction demand function upward.

4.2 Intensity of Social Influence

For notational simplicity, let us denote $z \equiv \frac{\lambda}{\beta+\kappa} > 0$, which measures the relative weight of social influence in driving consumer type change to that of idiosyncratic experience. The larger z, the higher weight social influence plays in changing consumer experience. Recall the optimal profit in equation (31). Let us perform the comparative statics over z:

$$\frac{\partial \Pi^*}{\partial z} = \frac{\partial I(\theta_1^*, z)}{\partial z}(1 - F(\theta_1^*)) \propto \frac{\partial}{\partial z}\left[\int_{\theta_1^*}^{\bar{\theta}} \frac{\left[\widehat{M}_1 - M_1(x)\right]}{1/z + \omega_D \widehat{M}_1 + \Delta\left[\widehat{M}_1 - M_1(x)\right]} dx\right] \tag{34}$$

Notice that the effect depends on the strength of upward or downward social influence. We further break down into the following two cases.

Stronger Upward Social Influence

Proposition 4. *When upward social influence is stronger than the negative one, as the intensity of social interactions z increases, the equilibrium profit $\Pi^*(\Delta > 0)$ increases, the marginal consumer θ_1^* lowers, the equilibrium price first drops then increases.*

So long as the social influence is positive, a more interactive digital consumption benefits the firm. Moreover, the firm should welcome more lower-valued consumers as θ_1^* decreases in z. Nonetheless, the impact of z on the equilibrium price is non-monotonic. To see this, let us further decompose the price effect:

$$\frac{\partial P(\theta_1^*, z)}{\partial z} = \underbrace{\left(1 + \frac{\partial I(\theta_1^*, z)}{\partial \theta_1^*}\right)}_{>0} \cdot \underbrace{\frac{\partial \theta_1^*}{\partial z}}_{<0} + \underbrace{\frac{\partial I(\theta_1^*, z)}{\partial z}}_{>0} \tag{35}$$

There are two opposite forces driving the equilibrium price. First, as social influence becomes more predominant, the firm is more likely to welcome more consumers with lower valuation to opt in, driving down the price. However, as z increases, the option value becomes higher. Hence, the equilibrium price is convex in z. The firm can charge a relatively higher price when digital consumption is either predominantly driven by idiosyncratic experience of by social interactions.

Stronger Downward Social Influence

When downward social influence is stronger, though the firm can be better off with social interactions than without, it is not clear how interactive digital consumption should be. The answer to this question becomes vital if we consider that the firm is able to moderate social interactions. We first present the result in the following proposition then explain the intuition.

Proposition 5. *When downward social interaction is stronger than the positive one, whether the equilibrium profit increases or decreases with the intensity of social interactions z depends on "social elasticity," i.e.,* $\frac{\partial \Pi^*}{\partial z} \gtrless 0$ *if* $\varepsilon_{SI} \equiv -\frac{\partial \ln\left[\widehat{M}_1 - M_1(x)\right]}{\partial \ln z} \lessgtr \frac{1}{1+z\omega_D \widehat{M}_1}$.

Our analysis hinges on the notion we coin as *"social elasticity"*, that is, the effect of one unit change in intensity of social interactions on the reduction in demand. Recall from Lemma 3, stronger downward social influence rotates the demand distribution downward, which reduces demand and hurts firm's profitability. And in this situation, the higher the intensity of social interactions, the larger amount the demand will reduce. Hence, it is key to understand the extent of demand loss in response to a unit increase in interaction intensity. Therefore, we define the concept of social elasticity, $\varepsilon_{SI} \equiv -\frac{\partial \ln\left[\widehat{M}_1 - M_1(x)\right]}{\partial \ln z}$ and find that the firm's profit can still increase with interaction intensity so long as the social elasticity is sufficiently small.

5 Extension

5.1 Firm's Product Quality

It becomes a strategic decision for the firm to design digital experience with the availability of social interactions. In this extension, we investigate the interplay between social interactions and firm's innate product quality.

Proposition 6. *Consider two underlying taste p.d.f f and g on the support $\left[\underline{\theta}, \overline{\theta}\right]$, which are ranked by monotone likelihood ratio:$\frac{f(\theta)}{g(\theta)}$ decreases in θ, we find that* $\Pi_F^* \leq \Pi_G^*$.

Only the upper tail of the underlying taste distribution (i.e., $\Pr_F(\theta \geq x) = 1 - F(x)$) affects the price function $P(\theta_1^*, z)$. Hence, it is intuitive to see that a distribution with more weight on its upper tail should be more welcome by the platform and delivers a higher level of profit to the platform. We formalize this comparative result in the above proposition. Here, the distribution g puts more weight on its upper tail (right tail) than the distribution f.

5.2 Purely Interactive Digital Consumption

With the burgeoning trend in live streaming on entertainment and gaming platforms such as Tiktok and Twitch, the firm is adapting to a more interactive community. In this extension, we examine the equilibrium paywall strategy when digital consumption becomes purely interactive.

Proposition 7. *When digital consumption is purely driven by social interactions, i.e., $z \to \infty$,*

- *When upward social influence is stronger than downward influence, $\theta_1^* = \underline{\theta}$, and $p^* = \overline{\theta}$;*
- *When upward social influence is equivalent to downward influence, $\theta_1^* = u_0$, and $p^* = E[\theta \geq 0] - u_0$;*
- *When downward social influence is stronger than upward influence, $\theta_1^* = \theta_{NI}$, and $p^* = p_{NI}$.*

The equilibrium paywall with purely interactive digital experience depends on the direction of social interactions. When the upward social influence is stronger, the firm should be as inclusive as possible. When the downward social influence is stronger, the firm's strategy reverts back to the no-interaction benchmark.

6 Conclusion

Social interactions play a crucial role in shaping consumers' digital content consumption experience and influencing their willingness to pay. Despite the prevalent and growing trend of in-consumption social interactions, firms confront novel challenges, including the loss of direct control over consumer experience and the uncertainty surrounding real-time interactions. Navigating these challenges becomes a critical concern, prompting questions about participation dynamics and the impact on paywall strategies. Our continuous-time model, incorporating idiosyncratic shocks and endogenous social interactions, reveals that firms benefit from allowing social interactions, even when downward influence outweighs upward influence. While the equilibrium price is lower, demand is higher compared to a no-interaction scenario. Importantly, in situations where downward social influence is stronger, the influence of social intensity on a firm's profitability hinges on the concept of "social elasticity." We demonstrate that a firm's profit can still rise with interaction intensity, provided the social elasticity remains sufficiently small. Our model framework and results can also be applied to the context of physical goods and provide managerial insights to firms that consider building interactive online customer communities.

Our research has a few limitations that we invite future researchers to address. Firstly, we do not consider social interactions between paid users and non-paid users. While social interactions are often provided for paid users in many contexts, future research can explore this possibility. Secondly, our work assumes that consumers always participate in social interactions. Nonetheless, their incentive can be further investigated, as we often observe consumer lurking.

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Appendix

Proof of Lemma 1

Proof. We first calculate the integral:

$$\begin{aligned}
\int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V\left(\theta'\right) dF\left(\theta'\right) &= \int_{\theta'=\underline{\theta}}^{\theta'=p+u_0} V_0 dF\left(\theta'\right) + \int_{\theta'=p+u_0}^{\theta'=\overline{\theta}} \left[V_0 + \frac{\beta}{\beta+\kappa}\left(\theta - p - u_0\right)\right] dF\left(\theta'\right) \\
&= V_0 + \frac{\beta}{\beta+\kappa}\int_{\theta'=p+u_0}^{\theta'=\overline{\theta}} \left(\theta - p - u_0\right) dF\left(\theta'\right) \\
&= V_0 + \frac{\beta}{\beta+\kappa}\int_{\theta=p+u_0}^{\theta=\overline{\theta}} \left[1 - F\left(\theta\right)\right] d\theta.
\end{aligned}$$

Plugging back into the expression of V_0, we obtain

$$V_0 = u_0 + \frac{\kappa}{\beta+\kappa}\int_{\theta=p+u_0}^{\theta=\overline{\theta}} \left[1 - F\left(\theta\right)\right] d\theta,$$

and

$$V_1\left(\theta\right) = \frac{\beta\left(\theta - p\right)}{\beta+\kappa} + \frac{\kappa}{\beta+\kappa}u_0 + \frac{\kappa}{\beta+\kappa}\int_{\theta=p+u_0}^{\theta=\overline{\theta}} \left[1 - F\left(\theta\right)\right] d\theta.$$

The type density function is given by $f\left(\theta\right)$. Since only those agents with $\theta \geq p + u_0$ participate, the total demand amounts to $1 - F\left(p + u_0\right)$. Hence, the profit is given by $p\left[1 - F\left(p + u_0\right)\right]$.

The price does not include option value but simply balance the instantaneous payoff from participating or not:

$$\theta_{NI} = p_{NI} + u_0.$$

It is direct to see that the LHS is strictly increasing and the RHS is strictly decreasing in p due to the log-concavity of $\left[1 - F\left(\theta\right)\right]$. Due to the following fact

$$\begin{aligned}
LHS\left(p = 0\right) &= 0 < \frac{1 - F\left(u_0\right)}{f\left(u_0\right)} = RHS\left(p = 0\right), \\
LHS\left(p = \overline{\theta} - u_0\right) &= \overline{\theta} - u_0 > 0 = RHS\left(p = \overline{\theta} - u_0\right),
\end{aligned}$$

we conclude that equation (9) implies a unique solution. □

Proof of Lemma 2

Proof. Noticing the fact $V_1(\theta_1^*) = V_0$, we obtain

$$V_1(\theta) = V_0 + \int_{x=\theta_1^*}^{\theta} \frac{\beta}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx. \tag{36}$$

Now,

$$\begin{aligned}
\int_{x=\underline{\theta}}^{\overline{\theta}} V(x)\, dF(x) &= \int_{x=\underline{\theta}}^{\theta_1^*} V_0 dF(x) + \int_{x=\theta_1^*}^{\overline{\theta}} V_1(x)\, dF(x) \\
&= V_0 + \int_{x=\theta_1^*}^{\overline{\theta}} [V_1(x) - V_0]\, dF(x) \\
&= V_0 - \int_{x=\theta_1^*}^{\overline{\theta}} [V_1(x) - V_0]\, d[1 - F(x)] \\
&= V_0 - \{[V_1(x) - V_0][1 - F(x)]\}|_{x=\theta_1^*}^{x=\overline{\theta}} + \int_{x=\theta_1^*}^{\overline{\theta}} [1 - F(x)] \frac{dV_1(x)}{dx} dx \\
&= V_0 + \int_{x=\theta_1^*}^{\overline{\theta}} \frac{\beta[1 - F(x)]}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx,
\end{aligned}$$

where we have used integral by parts.

Inserting this term into (7) yields

$$\begin{aligned}
(\beta + \kappa) V_0 &= \beta u_0 + \kappa \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V(\theta')\, dF(\theta') = \beta u_0 + \kappa V_0 + \kappa \int_{x=\theta_1^*}^{\overline{\theta}} \frac{\beta[1 - F(x)]}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx \\
&\Rightarrow \quad V_0 = u_0 + \int_{x=\theta_1^*}^{\overline{\theta}} \frac{\kappa[1 - F(x)]}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx. \qquad (37)
\end{aligned}$$

Besides, (14) can be simplified to

$$\begin{aligned}
\left(\beta + \kappa + \lambda\widehat{M}_1\right) V_1(\theta) &= \beta(\theta - p) + \kappa \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V(\theta')\, dF(\theta') \\
&\quad + \lambda\omega_D \int_{y=\theta_1^*}^{y=\theta} m_1(y) V_1(y)\, dy + \lambda(1 - \omega_D) V_1(\theta) M_1(\theta) \\
&\quad + \lambda\omega_U \int_{x=\theta}^{x=\overline{\theta}} m_1(x) V_1(x)\, dx + \lambda(1 - \omega_U) V_1(\theta) \left[\widehat{M}_1 - M_1(\theta)\right].
\end{aligned}$$

Setting $\theta = \theta_1^*$ in the above equation and noticing $M_1(\theta_1^*) = 0$ and $V_1(\theta_1^*) = V_0$, we obtain

$$\left(\beta + \kappa + \lambda\omega_U \widehat{M}_1\right) V_0 = \beta(\theta_1^* - p) + \kappa \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V(\theta')\, dF(\theta') + \lambda\omega_U \int_{x=\theta_1^*}^{x=\overline{\theta}} m_1(x) V_1(x)\, dx.$$

Then we can have

$$\lambda\omega_U \widehat{M}_1 V_0 = \beta(\theta_1^* - p - u_0) + \lambda\omega_U \int_{\theta_1^*}^{\overline{\theta}} m_1(x) V_1(x)\, dx. \tag{38}$$

We calculate the last integral as follows:

$$\begin{aligned}
\int_{\theta_1^*}^{\overline{\theta}} m_1(x) V_1(x)\, dx &= -\int_{\theta_1^*}^{\overline{\theta}} V_1(x)\, d\left[\widehat{M}_1 - M_1(x)\right] \\
&= -\left|\left\{V_1(x)\left[\widehat{M}_1 - M_1(x)\right]\right\}_{x=\theta_1^*}^{x=\overline{\theta}} + \int_{\theta_1^*}^{\overline{\theta}} \left[\widehat{M}_1 - M_1(x)\right] \frac{dV_1(x)}{dx} dx \right. \\
&= V_0 \widehat{M}_1 + \int_{\theta_1^*}^{\overline{\theta}} \frac{\beta\left[\widehat{M}_1 - M_1(x)\right]}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx.
\end{aligned}$$

Inserting back into (38) and rearranging,

$$\lambda\omega_U \widehat{M}_1 V_0 = \beta(\theta_1^* - p - u_0) + \lambda\omega_U \left[V_0 \widehat{M}_1 + \int_{\theta_1^*}^{\overline{\theta}} \frac{\beta\left[\widehat{M}_1 - M_1(x)\right]}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx\right],$$

we finally obtain an equation that associate θ_1^* with p:

$$p + u_0 = \theta_1^* + \int_{\theta_1^*}^{\overline{\theta}} \frac{\lambda\omega_U\left[\widehat{M}_1 - M_1(x)\right]}{\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(x)} dx, \tag{39}$$

which is exactly (18).

Verification. We want to show that the expression of $V_1(\theta)$ and V_0 so far obtained satisfy the defining equation of (14). To see this, we first rearrange this equation as

$$\begin{aligned}
\left[\beta + \kappa + \lambda\omega_U \widehat{M}_1 - \lambda\Delta M_1(\theta)\right] V_1(\theta) &= \beta(\theta - p) + \kappa \int_{\theta'=\underline{\theta}}^{\theta'=\overline{\theta}} V(\theta')\, dF(\theta') \\
&\quad + \lambda\omega_D \int_{y\in\Theta_1, y\le\theta} m_1(y) V(y)\, dy + \lambda\omega_U \int_{x\in\Theta_1, x\ge\theta} m_1(x) V(x)\, dx
\end{aligned} \tag{40}$$

We first calculate the integrals on the second line:

$$\begin{aligned}
\int_{y\in\Theta_1, y\leq\theta} m_1(y)\,V(y)\,dy &= \int_{\theta_1^*}^{\theta} m_1(y)\,V_1(y)\,dy = \int_{\theta_1^*}^{\theta} V_1(y)\,dM_1(y) = V_1(\theta)\,M_1(\theta) - \int_{\theta_1^*}^{\theta} M_1(y)\,\frac{dV_1(y)}{dy}dy \\
&= V_1(\theta)\,M_1(\theta) - \int_{\theta_1^*}^{\theta} \frac{\beta M_1(y)}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(y)}dy, \\
\int_{x\in\Theta_1, x\geq\theta} m_1(x)\,V(x)\,dx &= \int_{\theta}^{\overline{\theta}} m_1(x)\,V_1(x)\,dx = -\int_{\theta}^{\overline{\theta}} V_1(x)\,d\left[\widehat{M}_1 - M_1(x)\right] \\
&= V_1(\theta)\left[\widehat{M}_1 - M_1(\theta)\right] + \int_{\theta}^{\overline{\theta}} \left[\widehat{M}_1 - M_1(x)\right]\frac{dV_1(x)}{dx}dx \\
&= V_1(\theta)\left[\widehat{M}_1 - M_1(\theta)\right] + \int_{\theta}^{\overline{\theta}} \frac{\beta\left[\widehat{M}_1 - M_1(x)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx.
\end{aligned}$$

Inserting all these terms into the RHS yields

$$\begin{aligned}
&\beta(\theta-p) + \kappa V_0 + \int_{x=\theta_1^*}^{\overline{\theta}} \frac{\kappa\beta[1-F(x)]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx + \lambda\left[\omega_U\widehat{M}_1 - \Delta M_1(\theta)\right]V_1(\theta) \\
&- \int_{\theta_1^*}^{\theta} \frac{\lambda\omega_D\beta M_1(y)}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(y)}dy + \int_{\theta}^{\overline{\theta}} \frac{\lambda\omega_U\beta\left[\widehat{M}_1 - M_1(x)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx.
\end{aligned}$$

If the above is equal to the LHS of (40), we must have

$$\begin{aligned}
(\beta+\kappa)\,V_1(\theta) &= \beta(\theta-p) + \kappa V_0 + \int_{x=\theta_1^*}^{\overline{\theta}} \frac{\kappa\beta[1-F(x)]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx \\
&- \int_{\theta_1^*}^{\theta} \frac{\lambda\omega_D\beta M_1(y)}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(y)}dy + \int_{\theta}^{\overline{\theta}} \frac{\lambda\omega_U\beta\left[\widehat{M}_1 - M_1(x)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)}dx.
\end{aligned}$$

After substituting $V_1(\theta)$ given in (36) into the LHS and the expression of p given in (39) into the RHS, we will obtain the expression of V_0 as given in (37). □

Proof of Proposition 2

Proof. We can rewrite the platform's profit as follows:

$$[\theta_1^* - u_0 + I(\theta_1^*, z)]\,[1 - F(\theta_1^*)]$$

The F.O.C. condition w.r.t. θ_1^* gives

$$\theta_1^* - u_0 - \frac{1-F(\theta_1^*)}{f(\theta_1^*)} = -I(\theta_1^*, z) + \frac{dI(\theta_1^*, z)}{d\theta_1^*}\frac{1-F(\theta_1^*)}{f(\theta_1^*)}.$$

Recall that θ_{NI}, the marginal type of costumer in the benchmark case with no social interaction is implied by:

$$\theta_{NI} - u_0 - \frac{1-F(\theta_{NI})}{f(\theta_{NI})} = 0.$$

We have shown that $\frac{dI(\theta_1^*,z)}{d\theta_1^*} < 0$ and of course the option value is positive, so

$$\theta_1^* - u_0 - \frac{1-F(\theta_1^*)}{f(\theta_1^*)} < 0 = \theta_{NI} - u_0 - \frac{1-F(\theta_{NI})}{f(\theta_{NI})}.$$

Since $x - \frac{1-F(x)}{f(x)}$ is strictly increasing in x, we know

$$\theta_{NI} > \theta_1^*.$$

Knowing that $\frac{dp(\theta_1^*)}{d\theta_1^*} \in (0,1)$ and $\theta_1^* < \theta_{NI}$, we have the following to hold true:

$$p(\theta_1^*) < p(\theta_{NI}) < \theta_{NI} = p_{NI}.$$

We first compare the two integrands. According to (28), we know

$$\frac{\lambda\omega_U\left[\widehat{M}_1 - M_1(x)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1 - \lambda\Delta M_1(x)} \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} \frac{\lambda\omega_U\left[1-F(x)\right]}{\beta+\kappa+\lambda\omega_U\widehat{M}_1} \text{ if } \Delta \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} 0.$$

If we take θ_1^* as given, then the total measure of participants, $\widehat{M}_1 = 1 - F(\theta_1^*)$, is also fixed and we must have

$$P(\theta_1^*, \Delta) \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} P(\theta_1^*, 0) \text{ if } \Delta \left\{\begin{array}{c} > \\ = \\ < \end{array}\right\} 0. \tag{41}$$

The platform's profit can be written as

$$\Pi\left(\theta_1^*, \Delta\right) = P\left(\theta_1^*, \Delta\right)\left[1 - F\left(\theta_1^*\right)\right].$$

Denote by $\theta_1^*\left(\Delta\right)$ the optimal cutoff point for a general Δ, that is,

$$\theta_1^*\left(\Delta\right) = \arg\max_{\theta_1^*} \Pi\left(\theta_1^*, \Delta\right).$$

When $\Delta > 0$, we know

$$\begin{aligned} \Pi\left(\theta_1^*\left(\Delta\right), \Delta\right) &\geq \Pi\left(\theta_1^*\left(0\right), \Delta\right) = P\left(\theta_1^*\left(0\right), \Delta\right)\left[1 - F\left(\theta_1^*\left(0\right)\right)\right] \\ &> P\left(\theta_1^*\left(0\right), 0\right)\left[1 - F\left(\theta_1^*\left(0\right)\right)\right] = \Pi\left(\theta_1^*\left(0\right), 0\right), \end{aligned}$$

where the first inequality is because $\theta_1^*\left(\Delta\right)$ maximizes $\Pi\left(\theta_1^*, \Delta\right)$ and the second inequality is due to $P\left(\theta_1^*\left(0\right), \Delta\right) > P\left(\theta_1^*\left(0\right), 0\right)$ for $\Delta > 0$ given in (41).

The second part of the claim can be shown as follows,

$$\begin{aligned} \Pi^*(\Delta) &= \max_{\theta_1^*} P\left(\theta_1^*, \Delta\right)\left[1 - F\left(\theta_1^*\right)\right] \geq P\left(\theta_{NI}, \Delta\right)\left[1 - F\left(\theta_{NI}\right)\right] \\ &> p_{NI}\left[1 - F\left(\theta_{NI}\right)\right] = \Pi_{NI} \end{aligned}$$

When $\Delta < 0$, we know

$$\begin{aligned} \Pi\left(\theta_1^*\left(\Delta\right), \Delta\right) &= P\left(\theta_1^*\left(\Delta\right), \Delta\right)\left[1 - F\left(\theta_1^*\left(\Delta\right)\right)\right] < P\left(\theta_1^*\left(\Delta\right), 0\right)\left[1 - F\left(\theta_1^*\left(\Delta\right)\right)\right] = \Pi\left(\theta_1^*\left(\Delta\right), 0\right) \leq \Pi\left(\theta_1^*\left(0\right), 0\right) \\ &= P\left(\theta_1^*\left(0\right), 0\right)\left[1 - F\left(\theta_1^*\left(0\right)\right)\right]. \end{aligned}$$

where the first inequality is due to $P\left(\theta_1^*\left(\Delta\right), \Delta\right) < P\left(\theta_1^*\left(\Delta\right), 0\right)$ for $\Delta < 0$ given in (41) and the second inequality is because $\theta_1^*\left(0\right)$ maximizes $\Pi\left(\theta_1^*, 0\right)$.

$\square$

Proof of Proposition 4

Proof. We rewrite the price function as

$$P\left(\theta_1^*, \lambda\right) = \theta_1^* - u_0 + \int_{\theta_1^*}^{\overline{\theta}} \frac{\omega_U}{\frac{\beta+\kappa}{\lambda\left[\widehat{M}_1 - M_1(x)\right]} + \omega_D \frac{\widehat{M}_1}{\widehat{M}_1 - M_1(x)} + \Delta} dx.$$

The denominator of the integrand is strictly decreasing in λ, so the integrand as a whole is strictly increasing in λ. We have

$$\frac{\partial P\left(\theta_1^*, \lambda\right)}{\partial \lambda} > 0.$$

Secondly,

$$\frac{\partial P\left(\theta_1^*, \lambda\right)}{\partial \theta_1^*} = 1 - \frac{\omega_U}{\frac{\beta+\kappa}{\lambda \widehat{M}_1} + \omega_D + \Delta} = \frac{\beta+\kappa}{\beta+\kappa+\omega_U \lambda \widehat{M}_1},$$

so $\frac{\partial P\left(\theta_1^*\right)}{\partial \theta_1^*}$ is decreasing in λ:

$$\frac{\partial^2 P\left(\theta_1^*, \lambda\right)}{\partial \theta_1^* \partial \lambda} = -\frac{\beta+\kappa}{\left(\beta+\kappa+\omega_U \lambda \widehat{M}_1\right)^2} \omega_U \widehat{M}_1 < 0.$$

The FOC is given by

$$\frac{\partial P\left(\theta_1^*, \lambda\right)}{\partial \theta_1^*}\left[1 - F\left(\theta_1^*\right)\right] - f\left(\theta_1^*\right) P\left(\theta_1^*, \lambda\right) = 0.$$

Taking total differentiation wrt λ on both sides:

$$\frac{\partial FOC}{\partial \lambda} + \frac{\partial FOC}{\partial \theta_1^*} \frac{d\theta_1^*}{d\lambda} = 0,$$

where

$$\frac{\partial FOC}{\partial \theta_1^*} = \frac{\partial^2 P\left(\theta_1^*, \lambda\right)}{\partial \left(\theta_1^*\right)^2}\left[1 - F\left(\theta_1^*\right)\right] - 2\frac{\partial P\left(\theta_1^*, \lambda\right)}{\partial \theta_1^*} f\left(\theta_1^*\right) - f'\left(\theta_1^*\right) P\left(\theta_1^*, \lambda\right) \leq 0,$$

and

$$\frac{\partial FOC}{\partial \lambda} = \frac{\partial^2 P\left(\theta_1^*, \lambda\right)}{\partial \theta_1^* \partial \lambda}\left[1 - F\left(\theta_1^*\right)\right] - f\left(\theta_1^*\right) \frac{\partial P\left(\theta_1^*, \lambda\right)}{\partial \lambda} < 0.$$

We have

$$\frac{d\theta_1^*}{d\lambda} = -\frac{\frac{\partial FOC}{\partial \lambda}}{\frac{\partial FOC}{\partial \theta_1^*}} = -\frac{(-)}{(-)} < 0.$$

$\square$

Proof of Proposition 5

Proof. Following equation (34), we first show that

$$\frac{\partial}{\partial \lambda}\Pi\left(\theta_1^*, \lambda\right) \propto \frac{\partial}{\partial \lambda}\left\{\frac{\lambda\left[\widehat{M}_1 - M_1\left(x\right)\right]}{\beta + \kappa + \lambda\omega_D \widehat{M}_1 + \Delta\lambda\left[\widehat{M}_1 - M_1\left(x\right)\right]}\right\}$$

As shown earlier that the denominator is positive, we only need to show $\frac{\partial}{\partial \lambda}\left\{\lambda\left[\widehat{M}_1 - M_1\left(\theta\right)\right]\right\} > 0$. Note that the following equation holds:

$$\lambda\left[\widehat{M}_1 - M_1\left(\theta\right)\right] = \frac{1}{2}\left(\lambda\widehat{M}_1 - \frac{\kappa}{\Delta}\right) - \frac{1}{2}\sqrt{\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)^2 - 4\lambda\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]},$$

Therefore, we have

$$\begin{aligned}
\frac{\partial}{\partial \lambda}\left\{\lambda\left[\widehat{M}_1 - M_1\left(\theta\right)\right]\right\} &= \frac{1}{2}\widehat{M}_1 - \frac{1}{2}\frac{\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)\widehat{M}_1 - 2\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]}{\sqrt{\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)^2 - 4\lambda\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]}} \\
&= \frac{1}{2}\left[\widehat{M}_1 - \frac{\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)\widehat{M}_1 - 2\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]}{\sqrt{\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)^2 - 4\lambda\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]}}\right] > 0 \Leftrightarrow \\
\sqrt{\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)^2 - 4\lambda\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]} &> \left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right) - 2\frac{\kappa}{\Delta\widehat{M}_1}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right].
\end{aligned}$$

If the RHS is negative, then the inequality already holds. Otherwise, when the RHS is positive, we take square on both sides and obtain

$$\begin{gathered}
\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)^2 - 4\lambda\frac{\kappa}{\Delta}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right] > \left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right)^2 + 4\left(\frac{\kappa}{\Delta\widehat{M}_1}\right)^2\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]^2 \\
-4\frac{\kappa}{\Delta\widehat{M}_1}\left[F\left(\theta\right) - F\left(\theta_1^*\right)\right]\left(\lambda\widehat{M}_1 + \frac{\kappa}{\Delta}\right) \Leftrightarrow \widehat{M}_1 > F\left(\theta\right) - F\left(\theta_1^*\right),
\end{gathered}$$

which is already true.

$$P\left(\theta_1^*,\lambda\right)=\theta_1^*-u_0+\int_{\theta_1^*}^{\overline{\theta}}\frac{\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]\omega_U}{\beta+\kappa+\lambda\omega_D\widehat{M}_1+\Delta\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]}dx.$$

We have shown

$$Y\equiv\frac{\partial}{\partial\lambda}\left\{\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]\right\}=\left[\widehat{M}_1-M_1\left(x\right)\right]+\lambda\frac{\partial}{\partial\lambda}\left[\widehat{M}_1-M_1\left(x\right)\right]>0.$$

Hence, it follows that

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\Pi\left(\theta_1^*,\lambda\right) &\propto \frac{\partial}{\partial\lambda}\left\{\frac{\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]}{\beta+\kappa+\lambda\omega_D\widehat{M}_1+\Delta\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]}\right\}\\
&\propto \left\{\beta+\kappa+\lambda\omega_D\widehat{M}_1+\Delta\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]\right\}Y-\lambda\left[\widehat{M}_1-M_1\left(x\right)\right]\left(\omega_D\widehat{M}_1+\Delta Y\right)\\
&= \left(\beta+\kappa+\lambda\omega_D\widehat{M}_1\right)Y-\lambda\omega_D\widehat{M}_1\left[\widehat{M}_1-M_1\left(x\right)\right]\\
&= \left(\beta+\kappa\right)Y+\lambda\omega_D\widehat{M}_1\left(Y-\left[\widehat{M}_1-M_1\left(x\right)\right]\right)\\
&= \left(\beta+\kappa\right)Y+\lambda^2\omega_D\widehat{M}_1\frac{\partial}{\partial\lambda}\left[\widehat{M}_1-M_1\left(x\right)\right]\\
&= \left(\beta+\kappa\right)\left[\widehat{M}_1-M_1\left(x\right)\right]+\lambda\left(\beta+\kappa+\lambda\omega_D\widehat{M}_1\right)\frac{\partial}{\partial\lambda}\left[\widehat{M}_1-M_1\left(x\right)\right].
\end{aligned}$$

That is,

$$\frac{\partial}{\partial\lambda}\Pi\left(\theta_1^*,\lambda\right)\left\{\begin{array}{c}>\\=\\<\end{array}\right\}0\text{ iff }-\frac{\partial\ln\left[\widehat{M}_1-M_1\left(x\right)\right]}{\partial\ln\lambda}<\frac{\beta+\kappa}{\beta+\kappa+\lambda\omega_D\widehat{M}_1}.$$

□

Proof of Proposition 6

Proof. When the cutoff value θ_1^* is fixed, we have $P_F\left(\theta_1^*\right)\leq P_G\left(\theta_1^*\right)$ (where $P_F\left(\theta_1^*\right)$ is the price function under distribution F and $P_G\left(\theta_1^*\right)$ is defined in a similar way). Fixing the cutoff value θ_1^*, the price function associated with G is higher than that associated with F.

As for the profit comparision, we let Π_F^* be the optimized profit under distribution F:

$$\Pi_F^* = \max_{\theta_1^*} P_F(\theta_1^*)\left[1 - F(\theta_1^*)\right],$$

and Π_G^* is defined in a similar fashion:

$$\Pi_G^* = \max_{\theta_1^*} P_G(\theta_1^*)\left[1 - G(\theta_1^*)\right].$$

MLRP leads to several implications (all from Stochastic Orders and Their Applications by Shaked and Shanthikumar, 1994): (i) $\frac{f(\theta)}{1-F(\theta)} \leq \frac{g(\theta)}{1-G(\theta)}$ (ranked by hazard rate order, c.f. Theorem 1.C.1); (ii) $1 - F(\theta) \leq 1 - G(\theta)$ (hazard rate order implies first-order stochastic dominance, c.f. Theorem 1.B.1).

Let's define

$$Q_F(\theta_1^*) = \frac{\int_{\theta_1^*}^{\overline{\theta}}\left[1 - F(x)\right]dx}{1 - F(\theta_1^*)} = \int_{\theta_1^*}^{\overline{\theta}} \frac{xf(x)}{1 - F(\theta_1^*)}dx - \theta_1^* = E\left[x|x \geq \theta_1^*\right] - \theta_1^*,$$

then

$$Q_F(\theta_1^*) \leq Q_G(\theta_1^*).$$

Rewriting

$$P_F(\theta_1^*) = \theta_1^* - u_0 + \frac{Q_F(\theta_1^*)}{\frac{z}{1-F(\theta_1^*)} + 1},$$

we have

$$P_F(\theta_1^*) \leq P_G(\theta_1^*).$$

(This is because $\frac{z}{1-F(\theta_1^*)} \geq \frac{z}{1-G(\theta_1^*)}$ and $Q_F(\theta_1^*) \leq Q_G(\theta_1^*)$.) Fixing the cutoff value θ_1^*, the price function associated with G is higher than that associated with F: the distribution with more weight on the upper tail corresponds to a higher level of price.

Furthermore, let θ_F^* be the optimal cutoff value under taste distribution F:

$$\theta_F^* = \arg\max_{\theta_1^*} P_F(\theta_1^*)\left[1 - F(\theta_1^*)\right].$$

Define θ_G^* in a similar fashion. Define the associated profit for the platform as $\Pi_F^* = P_F(\theta_F^*)[1 - F(\theta_F^*)]$. Then we have the following chain of inequalities:

$$\Pi_F^* = P_F(\theta_F^*)[1 - F(\theta_F^*)] \overset{(a)}{\leq} P_G(\theta_F^*)[1 - F(\theta_F^*)] \overset{(b)}{\leq} P_G(\theta_F^*)[1 - G(\theta_F^*)] \overset{(c)}{\leq} P_G(\theta_G^*)[1 - G(\theta_G^*)] = \Pi_G^*,$$

where (a) is because $P_F(\theta_F^*) \leq P_G(\theta_F^*)$, (b) is because $1 - F(\theta_F^*) \leq 1 - G(\theta_F^*)$ and (c) is because θ_G^* optimizes the profit function. □

Proof of Proposition 7

Proof. We first derive $\theta_1^\infty = \lim_{\lambda\to\infty} \theta_1^*$. We first take $\lambda \to \infty$ in (19):

$$P(\theta_1^*, 0) = \theta_1^* - u_0 + \frac{\int_{\theta_1^*}^{\overline{\theta}} [1 - F(x)]\, dx}{1 - F(\theta_1^*)}.$$

Then the seller's objective function becomes

$$\begin{aligned} & (\theta_1^* - u_0)[1 - F(\theta_1^*)] + \int_{\theta_1^*}^{\overline{\theta}} [1 - F(x)]\, dx \\ = \quad & (\theta_1^* - u_0)[1 - F(\theta_1^*)] - \theta_1^*[1 - F(\theta_1^*)] + \int_{\theta_1^*}^{\overline{\theta}} x dF(x) \\ = \quad & -u_0[1 - F(\theta_1^*)] + \int_{\theta_1^*}^{\overline{\theta}} x dF(x) = \int_{\theta_1^*}^{\overline{\theta}} (x - u_0)\, dF(x)\,. \end{aligned}$$

FOC yields the desired result. It is direct to check that the SOC is also satisfied.

For sufficiently large λ, we conduct approximation as follows

$$\begin{aligned} 1 - F(\theta_1^*) &= 1 - F(u_0) - \frac{f(u_0)\,\theta_2^*}{\lambda} + o(1/\lambda)\,, \\ \int_{u_0 + \frac{\theta_2^*}{\lambda}}^{\overline{\theta}} [1 - F(x)]\, dx &= \int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx - \int_{u_0}^{u_0 + \frac{\theta_2^*}{\lambda}} [1 - F(x)]\, dx \\ &= \int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx - \frac{\theta_2^*}{\lambda}[1 - F(u_0)] + o(1/\lambda)\,, \end{aligned}$$

and the price

$$\begin{aligned} p &= \theta_1^* - u_0 + \frac{1}{\frac{\beta+\kappa}{\lambda\omega} + 1 - F(\theta_1^*)} \int_{\theta_1^*}^{\overline{\theta}} [1 - F(x)]\, dx \\ &= \frac{\int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx}{1 - F(u_0)} \left(1 - \frac{1}{\lambda} \frac{\frac{\beta+\kappa}{\omega} - f(u_0)\,\theta_2^*}{1 - F(u_0)}\right) + o(1/\lambda) \end{aligned}$$

We therefore obtain

$$\begin{aligned} \frac{dp}{d\theta_1^*} &= \frac{\frac{\beta+\kappa}{\lambda\omega}}{\frac{\beta+\kappa}{\lambda\omega} + 1 - F(\theta_1^*)} + \frac{f(\theta_1^*)}{\left[\frac{\beta+\kappa}{\lambda\omega} + 1 - F(\theta_1^*)\right]^2} \int_{\theta_1^*}^{\overline{\theta}} [1 - F(x)]\, dx \\ &= \frac{1}{\lambda} \frac{\beta+\kappa}{\omega[1 - F(u_0)]} + \frac{f(u_0) + \frac{f'(u_0)\theta_2^*}{\lambda}}{[1 - F(u_0)]^2} \left[1 - \frac{1}{\lambda} \frac{2\frac{\beta+\kappa}{\omega} - 2f(u_0)\,\theta_2^*}{1 - F(u_0)}\right] \left(\int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx - \frac{\theta_2^*}{\lambda} [1 - F(u_0)]\right) \\ &= \frac{f(u_0) \int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx}{[1 - F(u_0)]^2} + \frac{A_1}{\lambda} + o\left(\frac{1}{\lambda}\right), \end{aligned}$$

where

$$A_1 = \frac{\beta+\kappa}{\omega[1 - F(u_0)]} + \frac{f(u_0) \int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx}{[1 - F(u_0)]^2} \left(\frac{f'(u_0)\,\theta_2^*}{f(u_0)} - \frac{2\left[\frac{\beta+\kappa}{\omega} - f(u_0)\,\theta_2^*\right]}{1 - F(u_0)} - \frac{\theta_2^* [1 - F(u_0)]}{\int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx}\right). \tag{42}$$

Inserting all the above into the FOC (**??**) and matching the coefficients of terms of order $1/\lambda$ on both sides, we have

$$\frac{[1 - F(u_0)]^2}{f(u_0) \int_{u_0}^{\overline{\theta}} [1 - F(x)]\, dx} A_1 - \frac{f(u_0)\,\theta_2^*}{[1 - F(u_0)]} = \frac{\theta_2^* f'(u_0)}{f(u_0)} - \frac{\frac{\beta+\kappa}{\omega} - f(u_0)\,\theta_2^*}{1 - F(u_0)}.$$

Plugging the expression of A_1 given in (42) and rearranging, we obtain the the expression of θ_2^*.

We discuss the case for $\Delta \neq 0$ when $\lambda \to \infty$. Since the support of $M_1(\theta)$ is $\left[\theta_1^*, \overline{\theta}\right]$ and $\widehat{M}_1 = 1 - F(\theta_1^*)$, we have to study $\lim_{\lambda\to\infty} \theta_1^* \equiv \theta_1^\infty$. Rewriting the price function as follows:

$$P(\theta_1^*) = \theta_1^* - u_0 + \int_{\theta_1^*}^{\overline{\theta}} \frac{\omega_U \left[\widehat{M}_1 - M_1(x)\right]}{\frac{\beta+\kappa}{\lambda} + \omega_D \widehat{M}_1 + \Delta\left[\widehat{M}_1 - M_1(x)\right]} dx,$$

where

$$\widehat{M}_1 - M_1(\theta) = \begin{cases} \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\sqrt{\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}\left[F(\theta) - F(\theta_1^*)\right]}, \text{ if } \Delta > 0 \\ \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) - \frac{1}{2}\sqrt{\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}\left[F(\theta) - F(\theta_1^*)\right]}, \text{ if } \Delta < 0 \end{cases}.$$

When $\Delta < 0$, we have

$$\begin{aligned} \widehat{M}_1 &= \widehat{M}_1 - M_1(\theta_1^*) = \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) - \frac{1}{2}\sqrt{\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right)^2} \\ &= \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) - \frac{1}{2}\left|\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right| \\ &= \begin{cases} \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) - \frac{1}{2}\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right) = -\frac{\kappa}{\lambda\Delta}, \text{ if } \widehat{M}_1 + \frac{\kappa}{\lambda\Delta} > 0 \\ \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right) = \widehat{M}_1, \text{ if } \widehat{M}_1 + \frac{\kappa}{\lambda\Delta} < 0 \end{cases}. \end{aligned}$$

Hence, it has to be the case that $\widehat{M}_1 + \frac{\kappa}{\lambda\Delta} < 0$ when $\Delta < 0$.

When $\Delta > 0$, we have

$$\begin{aligned} 0 &= \widehat{M}_1 - M_1(\overline{\theta}) = \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\sqrt{\left(\widehat{M}_1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}\left[1 - F(\theta_1^*)\right]} \\ &= \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\sqrt{\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right)^2} = \frac{1}{2}\left(\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\left|\widehat{M}_1 - \frac{\kappa}{\lambda\Delta}\right| \\ &= \begin{cases} \widehat{M}_1 - \frac{\kappa}{\lambda\Delta}, \text{ if } \widehat{M}_1 > \frac{\kappa}{\lambda\Delta} \\ 0, \text{ if } \widehat{M}_1 < \frac{\kappa}{\lambda\Delta} \end{cases}. \end{aligned}$$

Hence, it has to be the case that $\widehat{M}_1 < \frac{\kappa}{\lambda\Delta}$ when $\Delta > 0$.

If $\theta_1^\infty < \overline{\theta}$, then $\lim_{\lambda\to\infty} \widehat{M}_1 = \widehat{M_1^\infty} > 0$ and therefore

$$\lim_{\lambda\to\infty}\left[\widehat{M}_1 - M_1(\theta)\right] = \begin{cases} \widehat{M_1^\infty}, \text{ if } \Delta > 0 \\ 0, \text{ if } \Delta < 0 \end{cases}.$$

This means that $M_1(\theta)$ has a mass at $\theta = \overline{\theta}$ when $\Delta > 0$ and has a mass at $\theta =$ when $\Delta < 0$. The price function under limit $\lambda \to \infty$ becomes

$$\lim_{\lambda\to\infty} P(\theta_1^*) = \begin{cases} \theta_1^\infty - u_0 + \int_{\theta_1^\infty}^{\overline{\theta}} \frac{\omega_U \widehat{M_1^\infty}}{\omega_D \widehat{M_1^\infty} + \Delta \widehat{M_1^\infty}} dx = \overline{\theta}, \text{ if } \Delta > 0 \\ \theta_1^\infty - u_0, \text{ if } \Delta < 0 \end{cases}$$

Hence, the profit optimization problem becomes

$$\max_{\theta_1^\infty < \overline{\theta}} P(\theta_1^\infty)\left[1 - F(\theta_1^\infty)\right] = \begin{cases} \overline{\theta}\left[1 - F(\theta_1^\infty)\right], \text{ if } \Delta > 0 \\ (\theta_1^\infty - u_0)\left[1 - F(\theta_1^\infty)\right], \text{ if } \Delta < 0 \end{cases}.$$

For $\Delta > 0$, the profit function is decreasing in θ_1^∞, so it attains its maximum at $\theta_1^\infty = \underline{\theta}$. and the platform should welcome all participants: $\widehat{M_1} = 1$. When λ is sufficiently large, θ_1^* already arrives at $\underline{\theta}$ such that $\widehat{M_1} = 1$ and

$$1 - M_1(\theta) = \frac{1}{2}\left(1 - \frac{\kappa}{\lambda\Delta}\right) + \frac{1}{2}\sqrt{\left(1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}F(\theta)}, \theta \in \left[\underline{\theta}, \overline{\theta}\right].$$

We find

$$\begin{aligned} \frac{\partial}{\partial\lambda}\left[1 - M_1(\theta)\right] &= \frac{1}{2}\frac{\kappa}{\lambda^2\Delta} + \frac{1}{4}\frac{-2\left(1 + \frac{\kappa}{\lambda\Delta}\right)\frac{\kappa}{\lambda^2\Delta} + 4\frac{\kappa}{\lambda^2\Delta}F(\theta)}{\sqrt{\left(1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}F(\theta)}} \\ &= \frac{\kappa}{2\lambda^2\Delta}\left(1 + \frac{-\left(1 + \frac{\kappa}{\lambda\Delta}\right) + 2F(\theta)}{\sqrt{\left(1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}F(\theta)}}\right) > 0 \Leftrightarrow \\ \sqrt{\left(1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}F(\theta)} &> -\left(1 + \frac{\kappa}{\lambda\Delta}\right) + 2F(\theta). \end{aligned}$$

If the RHS is negative, we already obtain the result. Otherwise, if the RHS is positive, we can take square on both sides and obtain

$$\begin{aligned} \left(1 + \frac{\kappa}{\lambda\Delta}\right)^2 - 4\frac{\kappa}{\lambda\Delta}F(\theta) &> \left(1 + \frac{\kappa}{\lambda\Delta}\right)^2 + 4\left[F(\theta)\right]^2 - 4\left(1 + \frac{\kappa}{\lambda\Delta}\right)F(\theta) \Leftrightarrow \\ 1 &> F(\theta), \end{aligned}$$

which already holds.

For λ sufficiently large, the price is given by

$$P_\lambda(\underline{\theta}) = \underline{\theta} - u_0 + \int_{\underline{\theta}}^{\overline{\theta}} \frac{\omega_U \left[\widehat{M}_1 - M_1(x)\right]}{\frac{\beta+\kappa}{\lambda} + \omega_D \widehat{M}_1 + \Delta\left[\widehat{M}_1 - M_1(x)\right]} dx.$$

Hence, $\frac{\partial P_\lambda(\underline{\theta})}{\partial \lambda} > 0$:

$$\frac{\partial P_\lambda(\underline{\theta})}{\partial \lambda} = \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial}{\partial \lambda} \frac{\omega_U [1 - M_1(x)]}{\frac{\beta+\kappa}{\lambda} + \omega_D + \Delta[1 - M_1(x)]} dx,$$

because

$$\frac{\partial}{\partial \lambda} \frac{\omega_U [1 - M_1(x)]}{\frac{\beta+\kappa}{\lambda} + \omega_D + \Delta[1 - M_1(x)]} = \frac{\partial}{\partial \lambda} \frac{\omega_U}{\frac{\beta+\kappa}{\lambda[1-M_1(x)]} + \frac{\omega_D}{1-M_1(x)} + \Delta} > 0.$$

Consequently, the profit is also increasing in λ because now the demand, fixed at 1, is irresponsive to λ and λ affects the profit only through $\frac{\partial P_\lambda(\underline{\theta})}{\partial \lambda} > 0$.

For $\Delta < 0$, the profit function is a product of price function (which is increasing in θ_1^∞) and total demand (which is decreasing in θ_1^∞). The FOC yields

$$\frac{1 - F(\theta_1^\infty)}{f(\theta_1^\infty)} - \theta_1^\infty + u_0 = 0.$$

Since $\frac{1-F(x)}{f(x)}$ is decreasing in x, the LHS is strictly decreasing in θ_1^∞. The above equation implies a unique solution if and only if

$$\begin{aligned} \frac{1}{f(\underline{\theta})} - \underline{\theta} + u_0 &> 0, \\ -\overline{\theta} + u_0 &< 0. \end{aligned}$$

The two inequalities are already held as we have assumed that $u_0 \in (\underline{\theta}, \overline{\theta})$. Obviously, we must have $\theta_1^\infty > u_0$.

Next, we show that the FOC implies the global optimization. For this, we first notice the following

$$\frac{d}{dx} P(x)[1 - F(x)] = f(x)\left[\frac{1 - F(x)}{f(x)} - (x - u_0)\right].$$

Hence,

$$sgn\left(\frac{d}{dx} P(x)[1 - F(x)]\right) = sgn\left[\frac{1 - F(x)}{f(x)} - (x - u_0)\right].$$

Since

$$\frac{1-F(x)}{f(x)}-(x-u_0)\begin{cases} >0, \text{ if } x<\theta_1^\infty \\ <0, \text{ if } x>\theta_1^\infty \end{cases},$$

it immediately follows that

$$\frac{d}{dx}P(x)\left[1-F(x)\right]\begin{cases} >0, \text{ if } x<\theta_1^\infty \\ <0, \text{ if } x>\theta_1^\infty \end{cases},$$

which implies that the profit function attains its global maximum at θ_1^∞.

Finally, we are interested in the asymptotic analysis for the case of $\Delta<0$.

For sufficiently large λ, the optimal cutoff can be expanded up to order as follows

$$\theta_1^*=\theta_1^\infty-\frac{\theta_2^\infty}{f(\theta_1^\infty)\lambda}+o(1/\lambda). \tag{43}$$

We use (43) to conduct approximation as follows

$$P(\theta_1^*)=\theta_1^*-u_0+\frac{\kappa}{-\lambda\Delta}\frac{\omega_U}{\omega_D\left[1-F(\theta_1^\infty)\right]^2}\int_{\theta_1^*}^{\bar{\theta}}\left[1-F(x)\right]dx,$$

$$\widehat{M}_1=1-F\left(\theta_1^\infty-\frac{\theta_2^\infty}{f(\theta_1^\infty)\lambda}\right)=1-F(\theta_1^\infty)+\frac{\theta_2^\infty}{\lambda}+o(1/\lambda),$$

$$\widehat{M}_1-M_1(\theta)=\frac{1}{2}\left[1-F(\theta_1^\infty)+\frac{1}{\lambda}\left(\theta_2^\infty-\frac{\kappa}{\Delta}\right)\right]-\frac{1}{2}\sqrt{\left[1-F(\theta_1^\infty)+\frac{1}{\lambda}\left(\theta_2^\infty+\frac{\kappa}{\Delta}\right)\right]^2-4\frac{\kappa}{\lambda\Delta}\left[F(\theta)-F(\theta_1^*)\right]}$$
$$=\frac{1}{2}\frac{\left[1-F(\theta_1^\infty)+\frac{1}{\lambda}\left(\theta_2^\infty-\frac{\kappa}{\Delta}\right)\right]^2-\left[1-F(\theta_1^\infty)+\frac{1}{\lambda}\left(\theta_2^\infty+\frac{\kappa}{\Delta}\right)\right]^2+4\frac{\kappa}{\lambda\Delta}\left[F(\theta)-F(\theta_1^*)\right]}{\left[1-F(\theta_1^\infty)+\frac{1}{\lambda}\left(\theta_2^\infty-\frac{\kappa}{\Delta}\right)\right]+\sqrt{\left[1-F(\theta_1^\infty)+\frac{1}{\lambda}\left(\theta_2^\infty+\frac{\kappa}{\Delta}\right)\right]^2-4\frac{\kappa}{\lambda\Delta}\left[F(\theta)-F(\theta_1^*)\right]}}+o(1/\lambda)$$
$$=\frac{\kappa}{-\lambda\Delta}\frac{1-F(\theta)}{1-F(\theta_1^\infty)}+o(1/\lambda),$$

and the price

$$\begin{aligned} P(\theta_1^*) &= \theta_1^\infty-\frac{\theta_2^\infty}{f(\theta_1^\infty)\lambda}-u_0-\frac{\omega_U}{\omega_D\left[1-F(\theta_1^\infty)\right]^2}\frac{\kappa}{\lambda\Delta}\int_{\theta_1^\infty-\frac{\theta_2^\infty}{f(\theta_1^\infty)\lambda}}^{\bar{\theta}}\left[1-F(x)\right]dx \\ &= \theta_1^\infty-u_0-\frac{1}{\lambda}\left[\frac{\omega_U}{\omega_D\left[1-F(\theta_1^\infty)\right]^2}\frac{\kappa}{\Delta}\int_{\theta_1^\infty}^{\bar{\theta}}\left[1-F(x)\right]dx+\frac{\theta_2^\infty}{f(\theta_1^\infty)}\right]+o(1/\lambda). \end{aligned}$$

The profit function is given by

$$\Pi(\lambda) = (\theta_1^\infty - u_0)\left[1 - F(\theta_1^\infty)\right] - \frac{1}{\lambda}\frac{\omega_U}{\omega_D\left[1 - F(\theta_1^\infty)\right]}\frac{\kappa}{\Delta}\int_{\theta_1^\infty}^{\overline{\theta}}\left[1 - F(x)\right]dx + o(1/\lambda),\ \Delta < 0$$

which is independent of θ_2^∞ up to the terms of orders no higher than $1/\lambda$. $\square$