Dynamic Price Competition: Theory and Evidence from Airline Markets*

Ali Hortaçsu
University of Chicago and NBER

Aniko Öry
Yale University

Kevin R. Williams†
Yale University and NBER

Preliminary and Incomplete: Please do not cite

June 2022

Abstract

We study a general dynamic pricing game where sellers are endowed with finite capacities and face uncertain demands toward a sales deadline. Price dynamics are determined not only by changing own-product opportunity costs and demand, but also by competitors' inventories as they affect future prices. We establish sufficient conditions for existence and uniqueness of equilibria, and for convergence to a system of differential equations. We then apply our framework to the airline industry using daily pricing and bookings data for competing airlines. We find that dynamic pricing results in higher output but lower total welfare than under uniform pricing.

*The views expressed herein are those of the authors and do not necessarily reflect the views of the National Bureau of Economic Research. We thank the anonymous airline for giving us access to the data used in this study. Under the agreement with the authors, the airline had "the right to delete any trade secret, proprietary, or Confidential Information" supplied by the airline. We agreed to take comments in good faith regarding statements that would lead a reader to identify the airline and damage the airline's reputation. All authors have no material financial relationships with entities related to this research. We thank Jose Betancourt for his excellent research assistance. We thank seminar participants at NYU and Yale for comments.

†Emails: hortacsu@gmail.com, aniko.oery@yale.edu, kevin.williams@yale.edu

1 Introduction

Dynamic pricing is commonly used by firms selling fixed inventory by a set deadline. Examples range from seats on airlines, trains and for entertainment events, to reservations for cruises, and inventory in retailing. In these markets, capacity influences prices in important ways. First, prices adjust as the opportunity cost of selling changes with scarcity—the value of a capacity unit depends on the ability to sell it in the future. Second, demand may change over time which provides the incentive to potentially hold inventory for certain customers. In competitive markets, including all of the aforementioned examples, not only are these forces present, but the opportunity cost of selling also depends on other firms' inventories because they affect future prices. What predictions can be made when competing firms dynamically adjust prices toward the deadline?

In this paper we study equilibria of a general dynamic pricing game and then apply our framework to the airline industry using granular data on competing airlines. We establish sufficient conditions for existence and uniqueness, and for convergence to a system of differential equations for an arbitrary number of firms and products. Our results show how little intuition from the well-studied single-firm setting carries over to competitive markets because of sellers' incentives to soften future price competition. For example, a firm may fire-sale units even if it has the smallest inventory in the industry in order to increase future prices. Or, a firm with excess capacity may charge high prices in order to get a competitor to sell out early. We then estimate the welfare effects of dynamic pricing in the airline industry using daily pricing and bookings data covering dozens of U.S. oligopoly markets. We estimate a model of air travel demand and show that dynamic pricing expands output, results in higher revenues, lowers consumer surplus, and decreases total welfare compared to uniform pricing. We examine alternative pricing regimes and show that not internalizing the scarcity of competitors can increase average prices in airline markets.

We begin by presenting a perfect information model for a duopoly where each firm offers a single product. In the appendix, we show that our results generalize to an arbitrary number of firms, each offering an arbitrary number of products. Each firm is exogenously endowed with limited initial capacity that must be sold by a deadline. After the deadline has passed, unsold capacities are scrapped with zero value. Firms are not allowed to oversell. Products are imperfect substitutes and satisfy general regularity conditions. Consumers arrive randomly according to time-varying arrival rates. Each consumer is short-lived and decides to purchase an available product or select an outside option, where the elasticity of demand can vary over time. Firms simultaneously choose prices after observing remaining capacities for all products; demand is realized, capacity constraints are updated, and the process repeats until the perishability date or until both products are sold out.

The model produces a rich set of equilibrium strategies because competitor prices affect both current market shares and opportunity costs of remaining capacity. We show that whether the incentive to soften future competition puts upward or downward pressure on a firm's price today depends on whether a sale of the competitor increases or decreases the firm's expected future profits. We call the change in continuation profits when the competitor sells, the "competitor scarcity effect." Typically, the competitor scarcity effect is negative, i.e., a sale of the competitor increases a firm's continuation value. In this case, higher demand for the competitor benefits the firm. This effect puts upward pressure on a firm's best-response in order to shift demand to its rival. We show that the competitor's price can even be a strategic substitute of the firm's price and demonstrate this finding with a commonly used demand function (discrete choice logit). Due to the competitor scarcity effect, firms' payoffs are also neither supermodular nor log-supermodular (Milgrom and Roberts, 1990). They are also not of the form studied in either Caplin and Nalebuff (1991) or Nocke and Schutz (2018). We show that the demand assumptions used in these papers do not guarantee uniqueness of equilibria in our model.

We derive sufficient conditions for existence and uniqueness of equilibria of the stage game using a theorem in Kellogg (1976). Although we show that even simple parametrizations of the model may yield multiple equilibria and price jumps, we prove that close to the deadline, our sufficient conditions for existence and uniqueness are always satisfied for commonly used demand systems.These conditions also ensure that the unique equilibrium price paths in the continuous-time limit of a discrete-time game are continuous and satisfy a system of differential equations. We demonstrate the usefulness of this characterization in our empirical analysis.

We prove that close the deadline, a sale of the firm with fewer units remaining increases future prices more than a sale of a firm with more units. The reason is that this sale significantly softens future price competition. The competitor scarcity effect for the firm with more units is large in magnitude and the own product scarcity effect is large for the firm with fewer products. Competition is fiercest when firms have the same number of units remaining. Thus, the ability to soften competition by strategically reacting to the distribution of remaining capacities raises prices in asymmetric states of the game. We use examples to show that internalizing the remaining capacity of a competitor can result in more or less price competition depending on consumer preferences.

Under the assumption of Independence of Irrelevant Alternatives (IIA), we can further derive a markup formula where the opportunity cost of selling for each firm consists of the own-product scarcity effect cost and the competitor scarcity effect, weighted by the equilibrium market share of the competitor relative to the outside option. This relative market share is independent of the firm's own price. We show that the firm's best-response problem given a competitor price parallels single-firm dynamic pricing models (Gallego and Van Ryzin, 1994), except that competitive forces are subsumed in the cost term. This allows us to decompose the drivers of pricing dynamics into a demand and opportunity cost effect.

In the second stage of our analysis, we quantify the welfare effects of dynamic price competition in the airline industry and study how different pricing regimes affects market outcomes. We use new data sources that contain daily pricing and booking data for competing airlines. The data are complete in that we observe all bookings (specifically, booking counts) for all nonstop competitors regardless of booking channel—tickets purchased either directly with the airline or via other agencies—for the routes studied.

We estimate a Poisson demand model, where aggregate demand uncertainty is captured through Poisson arrivals, and preferences are modeled through discrete choice nested logit demand. We use search data for one airline to inform arrival process parameters that are then scaled up to account for unobserved searches, e.g., via online travel agencies or other a competitor's website. In total, we estimate demand for 58 duopoly markets. We show there exists significant variation in willingness to pay both across routes and across days from departure for a given route. In general, demand becomes more inelastic as the departure date approaches. Average own-price elasticities are -1.4.

With the demand estimates, we first simulate equilibrium market outcomes using the differential equation characterization. This allows us to recover the own/competitor scarcity effects and firm strategies for all potential states—some games (departure dates) feature over 131 million potential states. Own scarcity effects are typically positive, and competitor scarcity effects are negative, i.e., a firm's continuation profit increases when the competitor sells. We find a very small percentage ($< 0.01\%$) of outcomes do not correspond to a game of strategic complements. We also show that if firms ignored the competitor scarcity effect throughout the game, prices and profits decrease—as predicted by the theory—and consumer surplus increases slightly. If firms additionally do not take into account their own scarcity effect, equilibrium prices and profits decline, and consumer surplus increases further. The net equilibrium welfare impact of ignoring scarcity effects is negative as more planes sellout in advance, before high-valuation consumers arrive. Therefore, incorporating scarcity dynamically increases allocative efficiency but also allows firms to extract more surplus in a competitive equilibrium.

We then simulate market outcomes when both firms use pricing heuristics that implement according to actual airline pricing practices. We consider two scenarios, one in which firms believe their competitor will charge the price observed last period, and the second where firms believe their competitor's price will follow a deterministic path over time. These pricing heuristics ignore that competitors are also responding to market outcomes dynamically. We find that the use of heuristics can result in higher or lower firm revenues, however, the total welfare effect is estimated to be positive compared to the benchmark model in both scenarios.

Finally, we simulate market outcomes if firms set a constant uniform price over time to examine the benefits and costs of dynamic pricing. Unlike what we know from single-firm analyses, we find that uniform pricing increases total welfare and benefits consumers compared to dynamic pricing, despite higher average posted prices. This indicates that dynamic pricing softens price competition despite featuring lower average prices due to extracting surplus from late-arriving, price insensitive customers.

1.1 Literature Review

This paper extends models of dynamic pricing with scarce inventory and a deadline (revenue management) to oligopoly (Gallego and Van Ryzin, 1994; Zhao and Zheng, 2000; Talluri and Van Ryzin, 2004). Several papers study the trade-offs when a single firm faces forward-looking buyers in a revenue management context (Board and Skrzypacz, 2016; Gershkov et al., 2018; Dilme and Li, 2019) Although we abstract from forward-looking buyers as previous work using airline clickstream data support short-lived buyers (Hortaçsu et al., 2021), we note that the intertemporal incentive to fire sale highlighted in these works also occurs in our model of competition.

We contribute to the literature on dynamic pricing competition, including Maskin and Tirole (1988); Dana (1999); Bergemann and Välimäki (2006); Sweeting et al. (2020). Both Dudey (1992) and Martínez-de Albéniz and Talluri (2011) consider price competition in the revenue management context with homogeneous consumers and symmetric firms. Our model allows asymmetric firms, arbitrarily differentiated products, and maps well to our empirical application of price competition in the airline industry. Dana and Williams (2022) consider an oligopoly model in which firms have an incentive to shift capacity to their rivals, but they abstract from demand uncertainty and differentiated products.

This paper adds a large literature on price dispersion in the airline industry by explicitly dynamic competition for a rich set of oligopoly markets (Borenstein and Rose, 1994; Stavins, 2001; Gerardi and Shapiro, 2009; Berry and Jia, 2010; Puller et al., n.d.; Sengupta and Wiggins, 2014; Siegert and Ulbricht, 2020). Our work also complements recent research on oligopoly pricing with algorithms (Calvano et al., 2020; Asker et al., 2021; Leisten, 2021; Hansen et al., 2021) in that we quantify market outcomes using heuristics that mimic industry practices.

2 Model of Dynamic Price Competition

We begin by detailing the demand assumptions that we use in our analysis (Section 2.1). Our exposition of demand is for an arbitrary number of products. In Section 2.2 we introduce supply-side notation by examining the single firm case. We then introduce a duopoly pricing game with two products in Section 2.3 which we analyze in Section 3. In Appendix A, we generalize these results to a pricing game with arbitrary number of firms and products using the notation of Sections 2.1-2.2.

2.1 Demand Model

We consider an economy with a set of products denoted by $\mathscr{J} := \{1, \ldots, J\}$. Products are imperfect substitutes and must be scrapped with zero value at a deadline $T > 0$. We analyze a discrete-time environment with periods $t \in \{0, \Delta, \ldots, T - \Delta\}$, $\Delta > 0$, and later consider the dynamics for the continuous-time approximation as $\Delta \to 0$. In every period, a consumer arrives with probability $\Delta\lambda_t$. Therefore, each consumer can be indexed by the time t of her arrival.

If all products are available and given a vector of prices $\mathbf{p} = (p_j)_{j \in \mathscr{J}}$, consumer t purchases product j with probability $s_j(\mathbf{p}; \boldsymbol{\theta}_t, \mathscr{J})$, where $\boldsymbol{\theta}_t \in \Theta \subset \mathbb{R}^n$ is a vector of $n > 1$ parameters that smoothly and deterministically depends on time t. This demand system satisfies the following regularity conditions.

Assumption 1. *i) Convergence for infinite prices: For any* j, $\lim_{p_j \to \infty} s_j(\mathbf{p}; \boldsymbol{\theta}, \mathscr{J}) = 0$.

For any subset $\mathscr{A} \subset \mathscr{J}$ *and* $j \in \mathscr{A}$ *the limit*[1]

$$s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A}) := \lim_{\substack{p_{j'}\to\infty \\ j'\notin\mathscr{A}}} s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J}) \in [0,1]$$

exists, where $p_{j'} = p_{j'}^{\mathscr{A}}$ *for all* $j' \in \mathscr{A}$, $\mathbf{p}^{\mathscr{A}} \in \mathbb{R}^{\mathscr{A}}$;

ii) Products are imperfect substitutes: For all $\mathscr{A} \subset \mathscr{J}$, $s_j(\mathbf{p};\boldsymbol{\theta},\mathscr{J})$ *is strictly decreasing in* p_j *and strictly increasing in* $p_{j'}$, $j' \neq j$;

iii) Differentiability and diagonally dominant Jacobi matrix of demand: For all $\boldsymbol{\theta}$ *and* $\mathscr{A} \subset \mathscr{J}$ *and* $j \in \mathscr{A}$, $s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})$ *is smooth in* $\mathbf{p}^{\mathscr{A}}$ *and*

$$\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})\right| > \sum_{j'\in\mathscr{A}\setminus\{j\}} \left|\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})\right|; \tag{1}$$

Furthermore, there exists a $C > 0$ *such that for all* $\mathbf{p}^{\mathscr{A}}$

$$|s_j(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})| < C \cdot \left(\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})\right| - \sum_{j'\in\mathscr{A}\setminus\{j\}} \left|\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p}^{\mathscr{A}};\boldsymbol{\theta},\mathscr{A})\right| \right). \tag{2}$$

Since products can sell out, we denote the set of available products in period t by $\mathscr{A}_t \subseteq \mathscr{J}$ and need to ensure all properties when prices become infinite. Condition (1) ensures by the Levy-Desplanques Theorem (see e.g. Theorem 6.1.10. in Horn and Johnson (2012)) that the Jacobi matrix $D_{\mathbf{p}}s(\cdot)$ is non-singular. The assumption also has an intuitive interpretation. A price change of product j should impact demand of product j more than it impacts the sum of demand of all other products. Condition (2) intuitively means that the demand for each product is bounded away from 1, and the differential impact of price changes is large relative to demand. It makes sure that optimal prices for a single firm and best responses in the oligopoly game are uniformly bounded given $\boldsymbol{\theta}$ and $\mathscr{A}$.

We omit the conditioning arguments in demand whenever the meaning is unambiguous.

[1] The limit is when all prices not in $\mathscr{A}$ are taken to infinity where the order does not matter.

Given Assumption 1, we can define for any $\mathscr{A}$ and finite price vector $\mathbf{p}$ the vector of inverse quasi own-price elasticities of demand as

$$\hat{\boldsymbol{\epsilon}}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}) := \left(D_{\mathbf{p}}\boldsymbol{s}(\mathbf{p};\boldsymbol{\theta},\mathscr{A})\right)^{-1}\boldsymbol{s}(\mathbf{p};\boldsymbol{\theta},\mathscr{A}).$$

Assumption 2 details the assumptions that we place on demand elasticities.

Assumption 2. *The system of inverse quasi own-price elasticity satisfies for all* $\boldsymbol{\theta}$ *and* $\mathscr{A} \subset \mathscr{J}$*:* $\det\left(D_{\mathbf{p}}\hat{\boldsymbol{\epsilon}}(\mathbf{p}) - I\right) \neq 0$ *for all* $\mathbf{p}$*, where* I *is the identity matrix.*

Assumption 2 guarantees that the system of first-order conditions has a unique solution. Given that Assumption 1 guarantees that $\max_{\mathbf{p}\in\mathbb{R}^{\mathscr{A}}} \mathbf{s}(\mathbf{p})(\mathbf{p}-\mathbf{c})$ has an interior solution, this replaces the assumption of log-concavity commonly made in single-product monopoly settings.

When the time index is relevant, we write $s_{j,t}(\mathbf{p}) := s_j(\mathbf{p};\boldsymbol{\theta}_t,\mathscr{A}_t)$. Further, we let the probability of choosing the outside option be equal to $s_{0,t}(\mathbf{p}) := 1 - \sum_{j\in\mathscr{J}} s_{j,t}(\mathbf{p})$.

We illustrate theoretical insights with a classic logit demand specification

$$s_{j,t}(\mathbf{p}) = \frac{\exp\left\{\frac{\delta_j - \alpha_t p_j}{\rho}\right\}}{1 + \sum_{j'\in\mathscr{A}_t} \exp\left\{\frac{\delta_{j'} - \alpha_t p_{j'}}{\rho}\right\}}, \tag{3}$$

where δ_j/ρ is the product-specific value of product j, α_t/ρ is the time-variant marginal utility to income, and $\rho > 0$ is a scaling factor. Note that when $\rho \to 0$, competition collapses to standard Bertrand. As $\rho \to \infty$, products become perfectly differentiated. Further, the empirical model in Section 5.1 uses a nested logit specification where the outside option of not buying is a separate nest from all inside goods. For this specification, we define

$$D_{\mathscr{A}_t} := \sum_{j\in\mathscr{A}_t} \exp\left\{\frac{\delta_j - \alpha_t p_j}{1-\sigma}\right\},$$

so that the probability that a consumer purchases j within the set of inside goods in period

t is equal to

$$s_{j\,|\,J,t} = \frac{\exp\left\{\frac{\delta_t - \alpha_t p_j}{1-\sigma}\right\}}{D_{\mathscr{A}_t}}.$$

It follows that the probability that a consumer purchases any inside good product is equal to

$$s_{\mathscr{A}_t} = \frac{D_{\mathscr{A}_t}^{1-\sigma}}{1 + D_{\mathscr{A}_t}^{1-\sigma}}.$$

Both classic logit and nested logit demand functions satisfy Assumptions 1 and 2 (see Appendix C).

2.2 Single Firm Benchmark

Before turning to competition, we first discuss a single firm, multi-product revenue management model with two goals in mind. The first is to introduce supply-side notation that we carry over to the competitive model. The second is to showcase that the single-firm problem is well behaved and exhibits nice properties. Most of these properties fail in the oligopoly model.

A single firm M offers J products for sale with an initial inventory $K_{j,0} \in \mathbb{N}$ of each product j. We do not model the initial capacity choice. Further, let $\mathbf{K}_t = (K_{j,t})_{j \in \mathscr{J}}$ denote the capacity vector at time t. The firm's continuation payoff at time $t < T$, given capacity vector $\mathbf{K} \in \mathbb{N}^{\mathscr{J}}$, satisfies the dynamic programming equation

$$\Pi_{M,t}(\mathbf{K};\Delta) = \\ \max_{\mathbf{p}} \Delta\lambda_t \sum_{j\in\mathscr{J}} s_{j,t}(\mathbf{p})\Big(p_j + \Pi_{M,t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)\Big) + \Big(1 - \Delta\lambda_t \sum_{j\in\mathscr{J}} s_{j,t}(\mathbf{p})\Big)\Pi_{M,t+\Delta}(\mathbf{K};\Delta),$$

where $\mathbf{e}_j \in \mathbb{R}^{\mathscr{J}}$ is a vector of zeros with a one in the jth position. The firm faces three boundary conditions: (i) $\Pi_{M,T+\Delta}(\cdot;\Delta) = 0$, (ii) $\Pi_{M,t}(\mathbf{0};\Delta) = 0$, where $\mathbf{0}$ is a vector of zeros, and (iii) $\Pi_{M,t}(\mathbf{K};\Delta) = -\infty$ if $K_j < 0$ for a $j \in \mathscr{J}$. These boundary conditions are simply stating that any remaining capacity is scrapped with zero value after the deadline T, and that the firm cannot obtain additional revenues after it sells all of its inventory. Note that the prices in period t do not directly affect the continuation values in period $t+\Delta$. Hence, the optimal price in each period solves a static maximization problem given the continuation payoffs. We denote this static profit-maximizing price vector by

$$\mathbf{p}_M(\boldsymbol{\omega}) := \arg\max_{\mathbf{p}} \sum_{j\in\mathscr{J}} s_j(\mathbf{p})\big(p_j - \omega_j\big),$$

where $\omega_j = \Pi_{M,t}(\mathbf{K};\Delta) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j;\Delta)$ is commonly called the *opportunity cost of selling product* j, and $\boldsymbol{\omega} = (\omega_j)_{j\in\mathscr{J}}$.[2] By Lemma 2 in Konovalov and Sándor (2010), Assumption 2-(iii) immediately implies that there is a unique optimal price vector which is continuous in $\boldsymbol{\omega}$ and $\boldsymbol{\theta}$. Then, by Lemma 11 in the Appendix, the continuous-time limit of this dynamic program exists, is unique, and solves the differential equation specified in the following lemma.

Lemma 1. $\Pi_{M,t}(\mathbf{K};\Delta)$ *converges uniformly to* $\Pi_{M,t}(\mathbf{K})$ *as* $\Delta \to 0$*, which solves the differential equation*

$$\dot{\Pi}_{M,t}(\mathbf{K}) = -\lambda_t \max_{\mathbf{p}} \sum_{j\in\mathscr{J}} s_{j,t}(\mathbf{p})\Big(p_j - \Big(\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j)\Big)\Big)$$

with boundary conditions $\Pi_{M,T}(\mathbf{K}) = 0$ *for all* $\mathbf{K}$*,* $\Pi_{M,t}(\mathbf{0}) = 0$*, and* $\Pi_{M,t}(\mathbf{K}) = -\infty$ *if* $K_j < 0$ *for a* j *for all* t*.*

Lemma 1 formalizes that the loss in continuation profit if no sale occurs is given by the forgone expected flow revenue $\lambda_t \max_{\mathbf{p}} \sum_{j\in\mathscr{J}} s_{j,t}(\mathbf{p})\big(p_j - (\Pi_{M,t}(\mathbf{K}) - \Pi_{M,t}(\mathbf{K}-\mathbf{e}_j))\big)$.

Given a capacity vector $\mathbf{K}$ and corresponding available products $\mathscr{A} = \{j : K_j \neq 0\}$, the first-order condition for the profit-maximizing prices $\mathbf{p}_{M,t}(\mathbf{K}) \in (p_j)_{j\in\mathscr{A}}$ can be written in matrix form

$$\mathbf{p}_{M,t}(\mathbf{K}) = \boldsymbol{\omega}_{M,t}(\mathbf{K}) - \big(D_{\mathbf{p}}\mathbf{s}_t(\mathbf{p}_{M,t}(\mathbf{K}))\big)^{-1}\mathbf{s}_t(\mathbf{p}_{M,t}(\mathbf{K})), \tag{4}$$

[2] Note that strictly speaking, the opportunity cost of selling product j is given by $\omega_j - \sum_{j'\neq j} \frac{s_j'(\mathbf{p})}{1-s_j(\mathbf{p})}\omega_{j'}$ as by selling product j, the firm forgoes the opportunity to sell any other product to the customer.

if $\boldsymbol{\omega}_{M,t}(\mathbf{K})$ are the opportunity costs of all products $j \in \mathscr{A}$ given capacity vector $\mathbf{K}$. Hence, the pricing policy $\mathbf{p}_{M,t}(\mathbf{K})$ is continuous in time and well behaved.

The evolution of the price vector $\mathbf{p}_{M,t}(\mathbf{K}_t)$ is then governed by the evolution of the random variable representing the opportunity costs and quasi-price elasticities of demand. The following proposition summarizes well-known properties of an optimal control problem, including monotonicity and concavity of the value function in the capacity vector. We also derive properties of the stochastic process governing the opportunity costs $\omega_{j,t}(\mathbf{K}_t)$.

Proposition 1. *The solution to the continuous-time single-firm revenue maximization problem in Lemma 1 satisfies the following:*

i) $\Pi_{M,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and increasing in* K_j *for all* $j \in \mathscr{J}$ *and* $t < T$*;*

ii) $\omega_{j,t}(\mathbf{K})$ *is decreasing in* t *for* $\mathbf{K} \neq \mathbf{0}$ *and decreasing in* K_j *for all* j *and* $t < T$*;*

iii) The stochastic process $\omega_{j,t}(\mathbf{K}_t)$ *is a submartingale.*

Statements i) and ii) of Proposition 1 simply say that continuation profits are increasing and concave in $\mathbf{K}$, and that continuation profits and opportunity costs are decreasing in t if $\mathbf{K}$ is held fixed. Statement (iii) implies that on average prices are increasing if $\boldsymbol{\theta}_t \equiv \boldsymbol{\theta}$ does not change over time by (4). This formal result has been shown in simulations, e.g., in McAfee and Te Velde (2006), where close to the deadline observed prices decrease since the infinite prices of sold out products are not taken into account.

2.3 Duopoly Model with Perfect Information

We introduce a duopoly pricing game with two firms $f \in \{1,2\}$. Each firm controls exactly one product, i.e., $\mathscr{J} = \{1,2\}$. Therefore, we set $j = f$ and use the subscript f to denote both the firm and product of interest. We generalize the results in this section to multiple firms with multiple products in Appendix A. Our exposition here focuses on the duopoly case with two products since this case is sufficient to highlight the key forces relevant for our analysis. Each firm f is initially endowed with $K_{f,0}$ units of its own product. In every period, firms simultaneously set prices $p_{f,t}$, and then a consumer arrives with probability $\Delta\lambda_t$. If a consumer arrives, she buys a product from firm f with probability $s_{f,t}(p_{1,t}, p_{2,t})$.

Like the single firm case, the payoff-relevant state is given by the vector of inventories $\mathbf{K} := (K_1, K_2)$ at time t. We are interested in Markov perfect equilibria in which each firm's strategy is measurable with respect to (K_1, K_2, t). We denote a Markov strategy of firm f by $p_{f,t}(\mathbf{K})$. Given equilibrium price vectors $\mathbf{p}_t^*(\mathbf{K}) := (p_{1,t}^*(\mathbf{K}), p_{2,t}^*(\mathbf{K}))$, firm f's value function satisfies

$$\begin{aligned}\Pi_{f,t}(\mathbf{K};\Delta) = \Delta\lambda_t \Bigg(&\underbrace{s_{f,t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\left(p_{f,t}^*(\mathbf{K}) + \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_f;\Delta)\right)}_{\text{revenue from own sale}} + \\ &\underbrace{s_{f',t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_{f'};\Delta)}_{\text{continuation value if } f' \text{ sells}}\Bigg) + \underbrace{\left(1-\Delta\lambda_t \sum_{h=\{1,2\}} s_{h,t}\left(\mathbf{p}_t^*(\mathbf{K})\right)\right)}_{\text{probability of no purchase}} \cdot \Pi_{f,t+\Delta}(\mathbf{K};\Delta),\end{aligned} \tag{5}$$

where we denote the competitor by $f' \neq f$. The boundary conditions are analogous to the single-firm case: (i) $\Pi_{f,t}(\mathbf{K};\Delta) = 0$ if $K_f = 0$, (ii) $\Pi_{f,t}(\mathbf{K};\Delta) = -\infty$ if $K_f < 0$, and (iii) $\Pi_{f,T+\Delta}(\mathbf{K};\Delta) = 0$ for all $\mathbf{K}$.

Similar to the single-firm setup, the period-t price vector does not impact the continuation payoffs in period $t+\Delta$. Hence, $p_t^*(\mathbf{K})$ is an equilibrium of a stage game in which firm f's payoff is given by $\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta)$. In order to describe this stage game, we denote for each firm $f \in \{1,2\}$ the change in continuation profit if product $h \in \{1,2\}$ by

$$\omega_{h,t}^f(\mathbf{K}) := \Pi_{f,t+\Delta}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K}-\mathbf{e}_h;\Delta),$$

which we call the *scarcity effect* of product h on firm f. If $h = f$, we call $\omega_{f,t}^f$ the *own-product scarcity effect*; when $h \neq f$, we call $\omega_{h,t}^f$ the *competitor scarcity effect*. We set $\omega_{f',t}^f := 0$ if $K_{f'} = 0$ for $f' \neq f$.[3] Further, we denote the matrix of scarcity effects that define

[3]We do not call the ωs opportunity costs for the same reason as discussed in Footnote 2.

the pricing stage game by

$$\Omega_t(\mathbf{K}) = \begin{pmatrix} \omega^1_{1,t}(\mathbf{K}) & \omega^1_{2,t}(\mathbf{K}) \\ \omega^2_{1,t}(\mathbf{K}) & \omega^2_{2,t}(\mathbf{K}) \end{pmatrix}.$$

In particular, it follows by Equation (5) that the flow payoff of firm f is equal to

$$\Pi_{f,t}(\mathbf{K};\Delta) - \Pi_{f,t+\Delta}(\mathbf{K};\Delta) = \Delta\lambda_t \left(s_{f,t}\left(\mathbf{p}^*_t(\mathbf{K})\right)\left(p^*_{f,t}(\mathbf{K}) - \omega^f_{f,t}(\mathbf{K})\right) - s_{f',t}\left(\mathbf{p}^*_t(\mathbf{K})\right)\omega^f_{f',t}(\mathbf{K})\right),$$

where $f' \neq f$.

In the stage game, firms simultaneously choose prices and receive payoffs

$$s_{f,t}(\mathbf{p})(p_f - \omega^f_{f,t}(\mathbf{K})) - s_{f',t}(\mathbf{p})\omega^f_{f',t}(\mathbf{K}),$$

where $f' \neq f$. Intuitively, the firm incurs an opportunity cost of selling its own product $\omega^f_{f,t}$ as in the single-firm setting, but future prices are also affected by the future degree of competition. This in turn is determined by the number of competitor units remaining. We use the following terminology in order to discuss intuition.

Definition 1. We say that a *competitor's sale intensifies competition* in a state $(\mathbf{K}, t)$ if $\omega^f_{f',t} > 0$ and that a *competitor's sale softens competition* in a state $(\mathbf{K}, t)$ if $\omega^f_{f',t} < 0$, where $f \neq f'$.

Due to how competition affects the stage game, we cannot apply results from Caplin and Nalebuff (1991) or Nocke and Schutz (2018). The payoffs are neither super-modular nor log-supermodular, so we cannot apply results from Milgrom and Roberts (1990). The stage game is also not a potential game. Therefore, we establish equilibrium properties of the game from scratch in the next section.

3 Analysis of a Duopoly Market

We derive theoretical properties of the dynamic pricing game under perfect information. We start with an analysis of uniqueness and continuity of equilibria. These properties are essential to be able to describe market outcomes effectively. We then discuss how demand and capacity realizations affect price levels and pricing dynamics. We highlight how endogenous asymmetries across firms can soften price competition.

3.1 Equilibrium Existence, Uniqueness, and Continuity

3.1.1 Sufficient Condition for Equilibrium Uniqueness in the Stage Game

We consider the stage game for an arbitrary matrix of opportunity costs Ω. We drop the time index and capacity argument in all expressions temporarily. Our first result presents sufficient conditions for existence and uniqueness of an equilibrium of the stage game using Lemma 2 (Kellogg, 1976) in Konovalov and Sándor (2010). To this end, we incorporate two additional assumptions that ensure equilibrium prices are the unique solution to the system of firms' first-order conditions.

We can write the first-order condition of firm f's profit maximization problem as

$$g_f(\mathbf{p}) = p_f,$$

where

$$g_f(\mathbf{p}) := \underbrace{\omega_f^f + \frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})}\omega_{f'}^f}_{\substack{\text{net opportunity cost} \\ \text{of selling}}} - \underbrace{s_f(\mathbf{p})\left(\frac{\partial s_f(\mathbf{p})}{\partial p_f}\right)^{-1}}_{\substack{\text{inverse quasi} \\ \text{own-price elasticity}}} . \tag{6}$$

Assumption 3 then guarantees that there is a unique solution to this system of equations.

Assumption 3. *i)* $\det\left(D_{\mathbf{p_f}}\big(\mathbf{g}_f(\mathbf{p})\big) - 1\right) \neq 0$ *for all* $\mathbf{p}$ *and* $f = 1, 2$.

ii) $\det\left(D_{\mathbf{p}}\big(\mathbf{g}(\mathbf{p})\big)-I\right)\neq 0$ *for all* $\mathbf{p}$*, where* $\mathbf{g}(\mathbf{p}) := (g_1(\mathbf{p}), g_2(\mathbf{p}))$.

To better understand Assumption 3, first note that with a single firm, the assumption guarantees that the first-order condition of the firm is either increasing or decreasing everywhere in its price. Assumption 3-(i) is always satisfied for demand functions that are log-concave in each dimension. Mathematically, Assumption 3 is related to Assumption 2, but the inverse quasi-own price elasticity is replaced by the function $\mathbf{g}(\mathbf{p})$. If the competitor scarcity effect is zero, Assumption 2 implies Assumption 3. If the competitor scarcity effect is not zero, the first-order condition is more complex than in the single-firm setting since the net opportunity cost of selling depends on the ratio of derivatives of the demand of the two firms. In general, this ratio depends on the firm's own price and the competitor's price.

Lemma 2. *Given Assumption 3 the stage game admits a unique equilibrium.*

Note that Lemma 2 establishes uniqueness and existence simultaneously. Under the commonly made assumption of independence of irrelevant alternatives (IIA) that is satisfied by a classic logit demand specification, we can independently establish existence of an equilibrium.

3.1.2 Continuity of Equilibrium Prices in Scarcity Effect Matrix Ω

In the dynamic game, the scarcity effect parameters Ω are endogenously changing over time. In this section, we parameterize the stage game by the scarcity effect and demand parameters, Ω and $\boldsymbol{\theta}$, respectively. We show that if Ω and $\boldsymbol{\theta}$ remain in a compact neighborhood in which the stage game admits a unique solution, then equilibrium prices denoted by $p^*(\Omega, \boldsymbol{\theta})$ are continuous in Ω and θ. Consequently, a small change in the opportunity costs does not change prices substantially. Hence, as long as the value functions do not jump, prices should not jump over time.

Lemma 3. *If the equilibrium of the stage game is unique for a compact set of* $(\Omega, \boldsymbol{\theta}) \in \mathscr{O}$*,* *then there exists an equilibrium price vector* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *for any* $(\Omega, \boldsymbol{\theta})$ *such that* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *is continuous in* $(\Omega, \boldsymbol{\theta})$ *on* $\mathscr{O}$.

Given Assumption 2, Assumption 3-ii) is satisfied for any matrix of opportunity costs Ω in a neighborhood $\mathscr{O}$ that contains the zero matrix $\Omega = \mathbf{0}$ (by continuity). In the next subsection we use this observation to show that close to the deadline in the continuous time limit, the price path converges to the solution of a system of differential equations (Lemma 9). Hence, in this region, price jumps occur only if one of the firms sells a unit.

Note, however, that for non-zero values of the opportunity costs, we can get multiplicities of equilibria that can potentially result in price jumps that are not caused by a change in inventory in the dynamic game. The following discussion illustrates this point. If firms' opportunity costs for the other firm are large, Assumption 3 can fail to hold. For example, with logit demand and $\delta_1 = \delta_2 = 0$, Assumption 3 is equivalent to

$$\left(s_1(\mathbf{p}) + \alpha \omega_2^1 s_0(\mathbf{p})\right)\left(s_2(\mathbf{p}) + \alpha \omega_1^2 s_0(\mathbf{p})\right) \neq 1 + \frac{1 - s_1(\mathbf{p}) - s_2(\mathbf{p})}{s_1(\mathbf{p}) s_2(\mathbf{p})}.$$

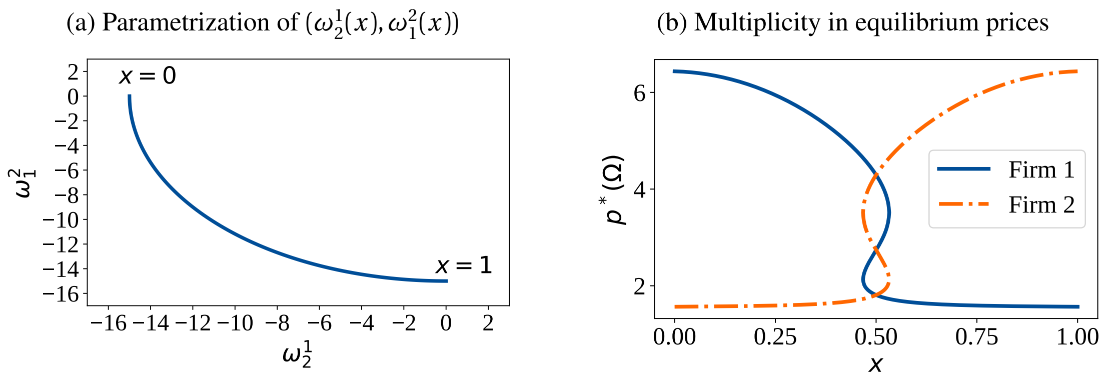
This condition does not depend on the firms' own-product scarcity effects ω_1^1 and ω_2^2 and is violated for $\mathbf{p} = \mathbf{0}$ and large positive competitor scarcity effects ω_2^1 and ω_1^2. The following parametrization of a curve of (ω_1^2, ω_2^1) illustrates how multiplicities can occur and we may observe jumps in prices even if the opportunity costs are changing continuously. Put differently, one can see in Figure 1-(b) that we cannot choose a smooth equilibrium price path of firm 1 as a function of x.

Consequently, close to the deadline T when $\Omega = \mathbf{0}$, the stage games are well behaved. However, further away from the deadline, when opportunity costs can potentially become large, the stage game may be less well behaved.

3.1.3 Equilibrium Dynamics

Lemma 4 (Continuous-time limit Limit)**.** *We assume that Assumptions 1, 2, and 3 are satisfied for* $\Omega = \mathbf{0}$. *For every* $\mathbf{K}$, *there exists a* $T_0(\mathbf{K}) > 0$, *non-increasing in* $\mathbf{K}$, *so that for*

Figure 1: Multiplicities in stage-game equilibria



Note: In these graphics we parametrize (ω_1^2, ω_2^1) with a curve $(\omega_2^1(x), \omega_1^2(x)) = \left(-15\cos\left(\frac{\pi}{2}x\right), -15\sin\left(\frac{\pi}{2}x\right)\right)$, $x \in [0,1]$, where we set $(\omega_1^1, \omega_2^2) = (0,0)$. Panel (a) depicts the parametrized curve and panel (b) equilibrium prices of both firms given (ω_1^2, ω_2^1) at varying values of x.

any $T \leq T_0(\mathbf{K})$ the value function $\Pi_{f,t}(\mathbf{K};\Delta)$ converges to a limit $\Pi_{f,t}(\mathbf{K})$ as $\Delta \to 0$ that solves the differential equation

$$\dot{\Pi}_{f,t}(\mathbf{K}) = -\lambda_t \quad \Big(s_{f,t}(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t)) \left(p_f^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t) - (\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_j)) \right) \\ - s_{f',t}(\mathbf{p}^*(\Omega_t(\mathbf{K}), \boldsymbol{\theta}_t)) \left(\Pi_{f,t}(\mathbf{K}) - \Pi_{f,t}(\mathbf{K} - \mathbf{e}_{f'}) \right) \Big),$$

where $f' \neq f$, with boundary conditions $\Pi_{f,T}(\mathbf{K}) = 0$ for all $\mathbf{K}$, $\Pi_{f,t}(\mathbf{K}) = 0$ if $K_f = 0$, and $\Pi_{f,t}(\mathbf{K}) = -\infty$ if $K_f < 0$.

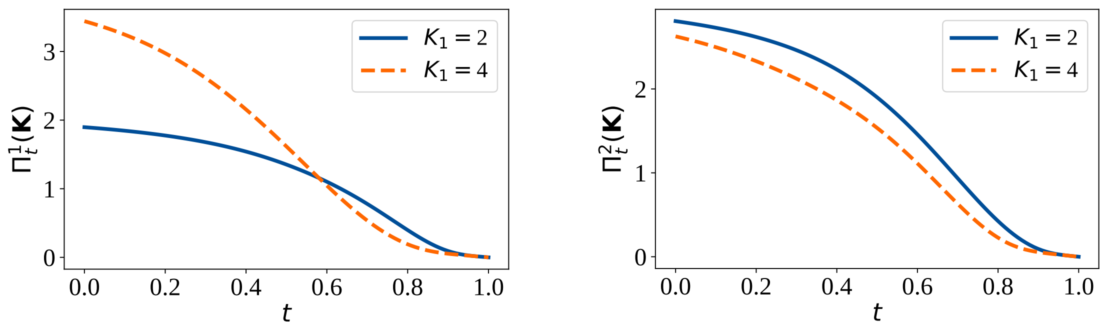
For the logit specification, we can additionally show that the convergence holds for any T as long as the scaling factor σ is sufficiently large, i.e., as long as products are sufficiently differentiated.

Lemma 5. *For the classic logit demand specification* (3)*, holding everything else fixed: there exists a $\bar{\rho}$ and a $\bar{\Delta} > 0$ so that for all $\rho > \bar{\rho}$ and $\Delta < \bar{\Delta}$, the cost matrix $\Omega_t(\mathbf{K})$ satisfies Assumption 4 for all $t \in [0, T]$ and $\mathbf{K} \leq \mathbf{K}_0$.*

The simulations in Figures 2 and 3 illustrate that none of the properties in Proposition 1 are generally satisfied in a duopoly. We fix the capacity of firm 2 to be $K_2 = 3$. First, Figure 2 shows that profits must not be monotonic in own capacity: close to the deadline, firm 1 expects higher profits with $K_1 = 2$ units than with $K_1 = 4$ units. In the following section, we discuss how this .

Figure 2: Simulated profits and prices for two firms ($K_2 = 3$)

(a) Firm 1 equilibrium profit $\Pi_t^1(K_1, 3)$ (b) Firm 2 equilibrium profit $\Pi_t^2(K_1, 3)$



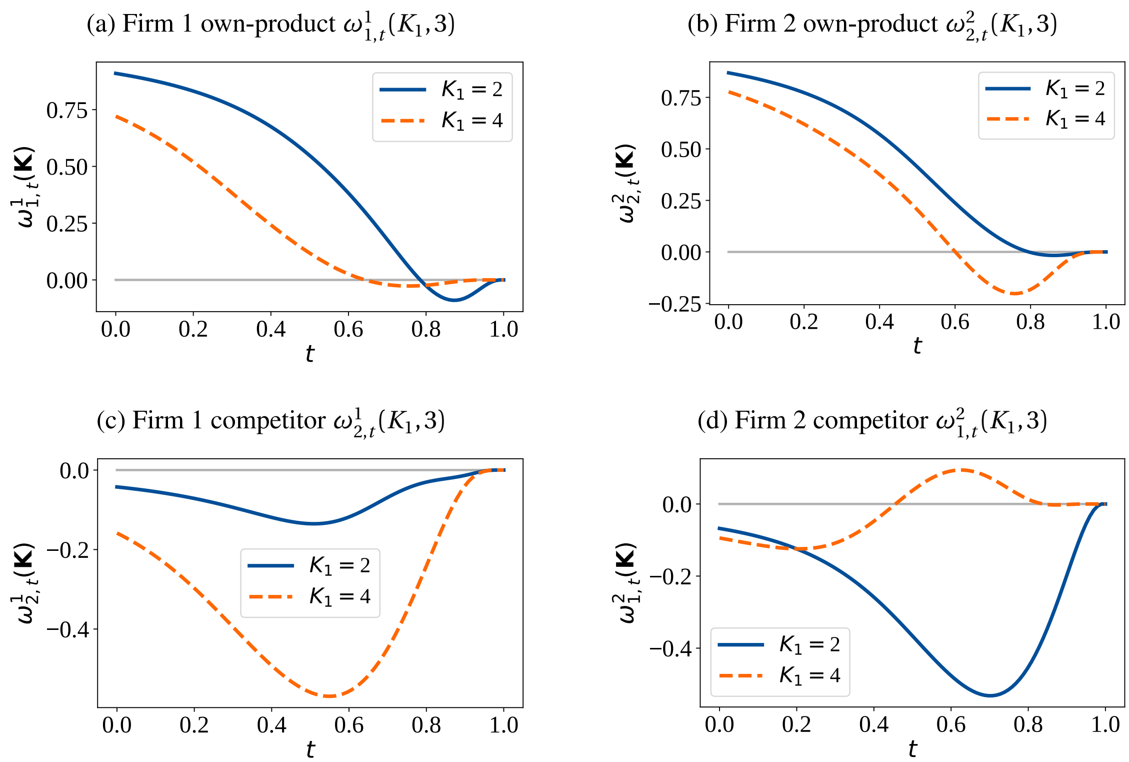
Notes: The simulations assume $\delta = (1, 1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

Figure 3 shows that all scarcity effect are non-monotonic in both time and capacities. It also appears that the competitor scarcity effects are large closer to the deadline, but become less important further away from the deadline.

3.2 Competition and Price Dynamics

Competition affects both price levels and pricing dynamics. In static Bertrand games, prices are strategic complements and hence, competition unambiguously lowers prices. These strategic forces can change in a game with positive competitor scarcity effects. This positive competitor scarcity effect can soften competition relative to a stage game with negative or zero competitor scarcity effect. We discuss this force in Section 3.2.1 using the logit demand specification. Then, in Section 3.2.2 we then show that these states of soft competition occur when the capacity is distributed asymmetrically, i.e., when one firm has many more units than another firm. Competition is strongest when both firms own the same number of units—even if firms are asymmetric with different mean consumer values, δ. Finally, we show that under the assumption of independence of irrelevant alternatives we can derive a mark-up formula that allows us to decompose how the dynamics of equilibrium prices is determined by the dynamics of the overall scarcity effect, the dynamics of the competitor scarcity effect, and the dynamics of demand elasticity. Additionally, we can show existence

Figure 3: Simulated opportunity costs for two firms ($K_2 = 3$)



(a) Firm 1 own-product $\omega^1_{1,t}(K_1, 3)$

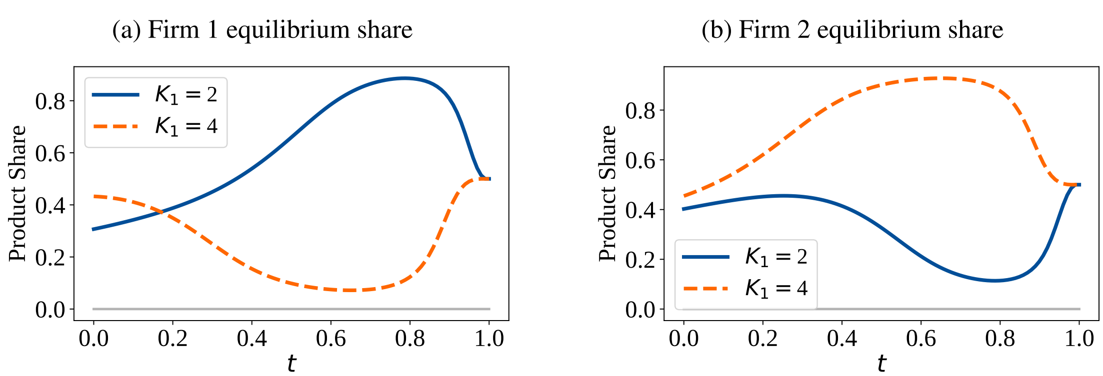
(b) Firm 2 own-product $\omega^2_{2,t}(K_1, 3)$

(c) Firm 1 competitor $\omega^1_{2,t}(K_1, 3)$

(d) Firm 2 competitor $\omega^2_{1,t}(K_1, 3)$

Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

Figure 4: Simulated market shares for two firms ($K_2 = 3$)



(a) Firm 1 equilibrium share

(b) Firm 2 equilibrium share

Notes: The simulations assume $\delta = (1,1)$, $\alpha_t \equiv 1$ and logit demand with scaling factor $\rho = 0.05$.

of equilibria in general under independence of irrelevant alternatives.

3.2.1 Prices as Strategic Substitutes

We have shown in Section 3.1.2 that multiplicity of equilibria and complex equilibrium dynamics are driven by the competitor scarcity effect. In this section we discuss further how exactly the competitor scarcity effect changes the strategic incentives in the stage game.

To this end, first consider the a stage game in which all competitor scarcity effects are set equal to zero, i.e., payoffs are of the form $s_f(\mathbf{p})(p_f - \omega_f^f)$.

When we analyze Bertrand games with such payoff functions, we usually assume log-concavity of demand to guarantee that the game is log-supermodular. In particular, we assume that the inverse quasi own price elasticity $\frac{s_f(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})}$ of each firm f given any price level of the competitor is increasing in the own price and decreasing in the competitor's price, i.e. demand becomes more inelastic if the competitor raises the price.[4] Such an assumption is satisfied by a logit or nested logit specification.

The competitor scarcity effect is typically negative, i.e., if the competitor loses a unit it benefits the firm.

To this end, consider the FOC of the game, given by $g_f(\mathbf{p}) = \mathbf{0}$ where g_f is defined in 6.

Consider the classic logit specification (3) with $\rho = 1$. Then, the weight on the competitor scarcity effect in (6) is given by

$$\frac{\frac{\partial s_{f'}}{\partial p_f}(\mathbf{p})}{\frac{\partial s_f}{\partial p_f}(\mathbf{p})} = -\frac{s_{f'}(\mathbf{p})}{1 - s_f(\mathbf{p})} = -\frac{\exp\left\{\delta_{f'} - \alpha p_{f'}\right\}}{1 + \exp\left\{\delta_{f'} - \alpha p_{f'}\right\}},$$

which is increasing in $p_{f'}$. Thus, if the competitor scarcity effect ω_2^1 is positive, an increase in competitor price increases a firm's cost of selling a product while for negative ω_2^1 the cost is decreasing in the competitor's price. Put differently, for negative ω_2^1, an increase in the competitor price puts downward pressure on a firm's own price. Therefore, the

[4] In Assumption 2, we only assume this to be true if the competitor price is infinity.

competitor's price can became a strategic substitute to the firm's own price, which differs from most pricing models where substitute products prices are strategic complements, i.e., the best response price increases in the competitor's price. Specifically, the first order condition of firm 1 can be written as

$$p_1 - \omega_1^1 + \frac{\exp\left\{\delta_2 - \alpha p_2\right\}}{1 + \exp\left\{\delta_2 - \alpha p_2\right\}} \omega_2^1 - \frac{1}{\alpha(1 - s_1(\mathbf{p}))} = 0$$

so by the implicit function theorem

$$\frac{\partial p_1^*}{\partial p_2} = \frac{s_2(\mathbf{p})}{1 + \exp\{\delta_2 - \alpha p_2\}} \left(\alpha \omega_2^1 + \exp\{\delta_1 - \alpha p_1\}\right)$$

Consequently, firm 2's price is a strategic substitute for firm 1's price if and only if

$$\omega_2^1 < -\frac{\exp\{\delta_1\}}{\alpha} < -\frac{\exp\{\delta_1 - \alpha p_1\}}{\alpha}.$$

Hence, if the price of firm $f' \neq f$ is a strategic substitute for firm f's price, then a sale of firm f' must soften competition. As can be seen in Figure 3, and as we show in the next subsection, this usually implies that a sale of firm f does not soften competition. In fact, we show next that close to the deadline competition is softens whenever the firm with the minimum capacity sells.

Finally note that this intuition of strategic substitutes and complements cannot be easily generalized to multi-product multi-firm competition. However, the insight in Proposition 2 is generally valid.

3.2.2 Capacity Distribution and Prices

In this section we investigate in which states the sale of a firm's own product or of the competitor's product can soften competition. To establish a formal result, we assume that λ_t and $s_{f,t}$ are independent of time, i.e., $\lambda_t = \lambda$, $\boldsymbol{\theta}_t = \boldsymbol{\theta}$. We do, however, allow for asymmetric, firm-specific demand s_f. We then show that close to the deadline prices can jump only after the sale of the firm with the lower capacity but not after a sale of the firm with more capacity.

Proposition 2. *Let $\lambda_t \equiv \lambda$, $\alpha_t \equiv \alpha$. Then, for $\mathbf{K}$ with $\underline{K} := \min_f K_f$, the following holds:*

$$p_{f,t}(\mathbf{K}) = p_{f,T}(\mathbf{K}) + \mathcal{O}(|T-t|^{\underline{K}}),\ t \to T \ \text{for}\ f = 1,2,$$

i.e., price changes close to the deadline are at most of order $\underline{K}$.
If $\lim_{t\to 0} \frac{\partial^{\underline{K}} \Pi_{h,t}}{(\partial t)^{\underline{K}}}(\mathbf{K}-\mathbf{e}_h) \neq 0$ *for all f with $K_h = \underline{K}$, then*[5]

$$p_{f,t}(\mathbf{K}) = p_{f,T}(\mathbf{K}) + \Theta(|T-t|^{\underline{K}}),\ t \to T \ \text{for}\ f = 1,2,$$

i.e., price changes are exactly of order $\underline{K}$.

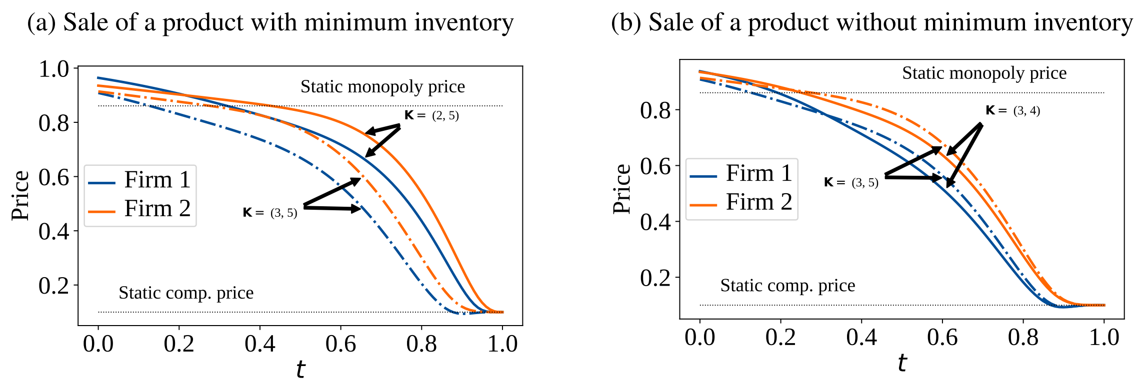
Formally these statements say that the order of change of the price over time is determined by the minimum capacity of products in the market. The order of the slope of prices as they converge to the static competitive price only changes if the minimum capacity in the market changes. The simulation in Figure 5 of price policy changes illustrates the impact of a sale of the product with minimum inventory versus the sale of a product which has not minimal inventory. We use similar parameters as before, $\delta_1 = \delta_2 = 1$ and sales starting with $\mathbf{K} = (3,5)$. Panel (a) shows that if firm 1 with less capacity sells, all prices jump up. However, in panel (b) one can see that a sale of firm 2 does not change prices much.

The prospect of significant price changes goes hand-in-hand with the desire to have the firm with minimum capacity sell. Put differently, only a sale of the firm with less capacity softens price competition significantly. In the extreme case, if firms have the same capacity, then any sale leads to a price jump regardless of who sells. Then, both firms would like to leave this state of fierce competition as soon as possible by offering low prices — possibly even prices smaller than the competitive price.

Empirically, this means that we expect that firms benefit whenever remaining capacities

[5] Recall that $f(t) = \mathcal{O}(g(t))$ as $t \to T$ if $\exists \delta, C_1 > 0$ so that for all t with $0 < |T-t| < \delta$, $|f(t)| \leq C_1 g(t)$. $f(t) = \Theta(g(t))$ if additionally $\exists C_2 > 0$ so that $C_2 g(t) \leq |f(t)|$.

Figure 5: Price paths before and after a sale



Notes: These simulations correspond to the parameter values $\delta_j = 1$, $\alpha = 1$, $\sigma = 0.05$ and $\lambda = 10$.

are distributed unequally across firms. This is because the state of the market inventory can serve as a coordination device for firms to decide who should sell their inventory first.

3.2.3 Independence of Irrelevant Alternatives and Markup formula

For demand specifications that satisfy the commonly used assumption of "Independence of Irrelevant Alternatives (IIA)," we can establish existence more generally and derive an economically meaningful markup formula where the net opportunity cost of selling is independent of a firm's own price.

Assumption 4 (Independence of Irrelevant Alternatives (IIA))**.**

$$\frac{\partial}{\partial p_1}\frac{s_2(\mathbf{p})}{s_0(\mathbf{p})} = \frac{\partial}{\partial p_2}\frac{s_1(\mathbf{p})}{s_0(\mathbf{p})} = 0.$$

Given Assumptions 1, 2 and 4, we establish the following proposition: [6]

Proposition 3 (Mark-up formula under IIA)**.** *Under Assumptions 1,2, and 5, there exist functions* $d_1(p_2;\boldsymbol{\omega},\alpha), d_2(p_1;\boldsymbol{\omega},\alpha)$ *so that the equilibrium prices of the stage game coincide*

[6] The general result in Appendix A additionally shows that with multiple products for each firm, the game can be transformed to a game in which each product is managed by its own firm given transformed payoff functions.

with the equilibrium prices of a game where firms simultaneously choose their prices p_f maximizing

$$\begin{cases} s_1(\mathbf{p})(p_1 - c_1(p_2; \boldsymbol{\omega}, \alpha)) + d_1(p_2; \boldsymbol{\omega}, \alpha) \\ s_2(\mathbf{p})(p_2 - c_2(p_1; \boldsymbol{\omega}, \alpha)) + d_2(p_1; \boldsymbol{\omega}, \alpha) \end{cases}$$

where $f \neq -f$ and

$$c_1(p_2; \boldsymbol{\omega}, \alpha) := \omega_1^1 + \tilde{s}_2(p_2)\omega_2^1, \qquad c_2(p_1; \boldsymbol{\omega}, \alpha) := \omega_2^2 + \tilde{s}_1(p_1)\omega_1^2. \tag{7}$$

where $\tilde{s}_2(p_2) := \frac{s_2(\mathbf{p})}{s_0(\mathbf{p})}$ and $\tilde{s}_1(p_1) := \frac{s_1(\mathbf{p})}{s_0(\mathbf{p})}$ is the demand of firms 2 and 1, respectively, conditional on the other firm not selling.

A consequence of Proposition 3 is that the first-order conditions (FOCs) that implicitly define the best-response functions of the firms can be written in a markup formulation as

$$\frac{p_f - c_f(p_{-f}; \boldsymbol{\omega}, \alpha)}{p_f} = -\frac{1}{\epsilon_f(\mathbf{p})}, \tag{8}$$

where $\epsilon_f(\mathbf{p}) = \frac{\partial s_f(\mathbf{p})}{\partial p_f} \frac{p_f}{s_f(\mathbf{p})}$ is the elasticity of demand. Equation (8) shows that the price dynamics is governed by the dynamic evolution of the net opportunity cost c_f and the change in demand elasticity, and the evolution of the net opportunity cost depends on both the own-product scarcity effect and the competitor scarcity effect—weighted by the relative market share of the competitor relative to the outside option. Thus, if many consumers pick the outside option, or if the competitor is small, a firm's decision is not much affected by the competitor. In turn, if the competitor is large, the competitor scarcity effect has a larger weight.

Finally, IIA also automatically guarantees existence of equilibria.

Lemma 6 (Existence)**.** *Assume that Assumptions 1, 2,* **??***, and 4 are satisfied. Then, there exists an equilibrium to the above stage game for any cost matrix Ω.*

3.2.4 Pricing with Heuristics

We compare the benchmark model to two pricing heuristics where firms do not internalize the scarcity of their competitor. Moreover, firms do not explicitly account for the fact that their competitor is a strategic agent solving a dynamic pricing problem. In both heuristics, we consider discrete prices as they are used in actual airline pricing practices. Applied theory work, including Asker et al. (2021), also consider discrete prices. The pricing menu (set of discrete prices for all time periods) is taken as given.

We label the heuristics "Lagged Model" and "Deterministic Model," respectively. In the lagged model, each firm, having observed its competitor's last period price, assumes this price will also be charged in the current and all future periods. Each firm then calculates its residual demand curves in all remaining periods and solves a single-firm dynamic programming problem. In the deterministic model, each firm simply assumes its competitor will price at the lowest possible price in all remaining periods.

We present price path simulations of the heuristics in Appendix Figure 15.

4 Data and Descriptive Evidence

4.1 Data Description

Our empirical insights are derived from data provided to us through a research partnership with a large U.S. airline.[7] The core data set contains booking and pricing information on competing airlines and was assembled by third parties that collect and combine contributed data. The data have strong parallels with other contributed data sets, such as the the Nielsen scanner data used to study retailing, in that we observe prices and quantities for competing firms.

The bookings data track flight-level sales counts over time. We use the tuple j, t, d to denote an airline-flight number, day before departure, departure date combination. The

[7] The airline has elected to remain anonymous.

frequency of the data is daily. We observe separate booking counts for passengers flying between an origin-destination pair (OD) and consumers making connections. We call these consumers *local* and *flow* passengers, respectively. Our structural analysis focuses on local, nonstop traffic. We do not model the potential for consumers to connect while flying between an origin-destination pair.

We observe bookings for consumers who purchased directly with the airline and on other booking channels, e.g., online travel agencies. We label these bookings *direct* and *indirect*, respectively. Because we observe all booking counts, we can construct the load factor for each flight over time. We do not know the exact itinerary involved for each booking, e.g., a round-trip versus a one-way booking. Therefore, we assume that the price paid for each nonstop booking corresponds to the lowest available nonstop, one-way fare for that flight.

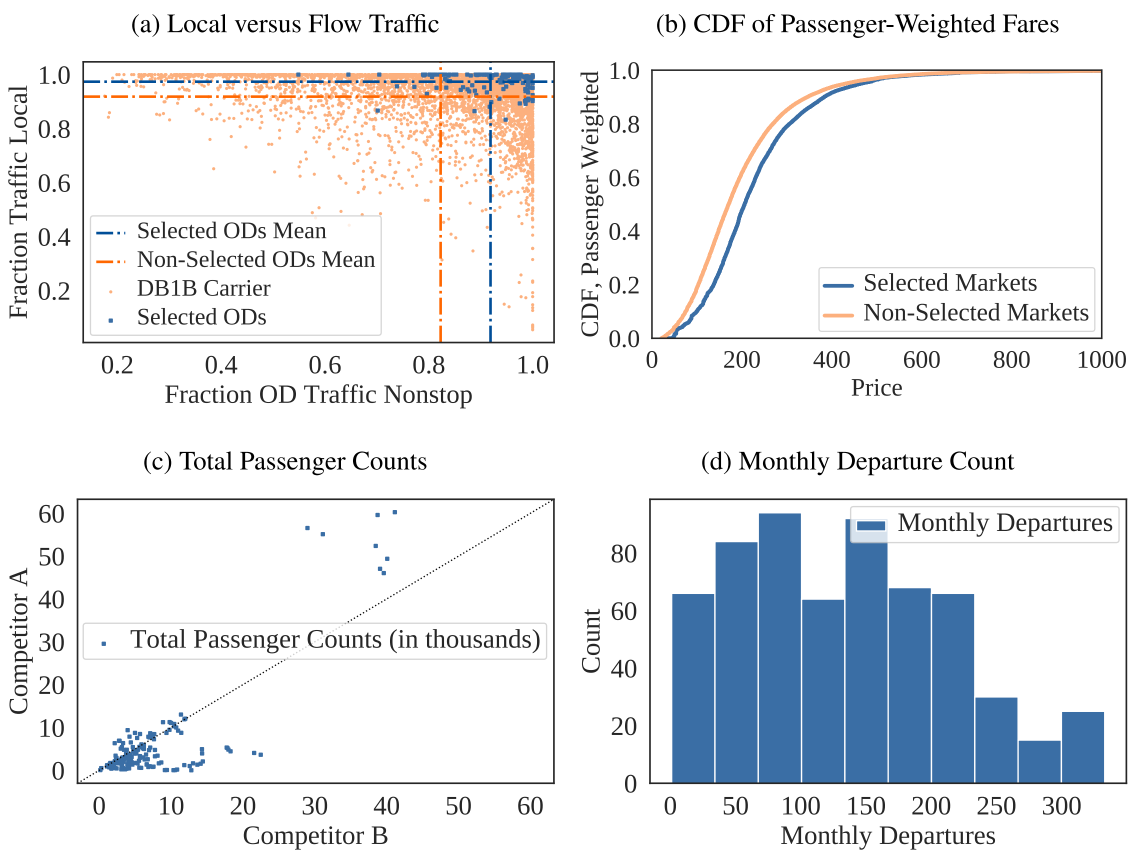
Our pricing data come from a separate third-party data provider that gathers and disseminates fare information for the airline industry. The data frequency matches the booking information, i.e., we observe daily prices at the flight level. We observe fares even when there are no bookings. Several prices are tracked, including tickets of different qualities (cabins, fully refundable, etc.). We concentrate our analysis on the lowest available economy class ticket because travelers overwhelmingly purchase the lowest fare offered (Hortaçsu et al., 2021). We do not model consumers choosing between cabins (economy vs. first class) nor the pricing decision for different versions of tickets.

In order to gauge market sizes, we use clickstream search data provided to us by the air carrier. See Hortaçsu et al. (2021) for more details. Observed searches understate true arrivals because some consumers may search and purchase through online travel agencies or directly with competitors. We extrapolate total arrivals by scaling up observed searches using hyperparameters that we describe below.

4.2 Route Selection

Our analysis concentrates on nonstop flight competition. We limit ourselves to routes where nonstop service is provided by exactly two airlines—by our data provider and one competitor. Our data contain more than one competitor airline, however, we will always refer to the competing airline as "the competitor." We eliminate routes where the third-party data is incomplete, e.g., where a carrier provides direct bookings to the data provider but indirect bookings are missing. In addition to these criteria, we select routes in which most OD traffic is traveling nonstop. This selection criteria allows us to avoid the additional complexity of modeling connecting traffic.

Figure 6: Summary Analysis from the DB1B Data



Note: Panel (a) records the percentage of flow (connecting) vs local traffic and the percentage of non-stop traffic in the DB1B data. Panel (b) plots the cdf of prices for selected routes and all dual-carrier markets. Panel (c) reports total passenger counts for both competitors. Panel (d) reports the number of aggregate monthly departures for the routes in our sample.

In Figure 6 we provide summary analysis of the 58 routes in our data using the publicly available DB1B data. These data contain 10% of bookings in the U.S. but lack information on the booking and departure date. In panel (a), we show the percentage of total traffic that is local versus the percentage of local traffic flying nonstop for our data compared to all dual-carrier nonstop markets in the U.S. The selected markets primarily contain local traffic that are traveling nonstop. In panel (b) we show that the distribution of fares in our markets is similar to the universe of dual-carrier markets.

In panels (c) we use the DB1B data to compute the quarterly passenger counts of the competing airlines in our data set. The panel shows the total passenger count for "Competitor A" and "Competitor B," which we use to denote our air carrier and the nonstop competitor, respectively. Each dot represents an OD-quarter. The panel shows the diversity of routes in our sample. There is considerable variation in the total size of the market (distance from the origin) as well as the relative size of the airlines for each OD. There is also variation in the passenger count of nonstop traffic within an OD across carriers.

Finally, in panel (d) we use the publicly available T100 segment data to plot the total number of monthly departures for the routes in our sample. Over half of our sample contains routes in which there are less than five daily frequencies between the origin and destination. Several routes feature twice daily service (one flight per airline). At the other extreme, one route in our data contains nearly 10 daily frequencies.

4.3 Descriptive Evidence

We provide a summary of the main data in Table 1. Average fares across airlines in our sample are $233. On average, each flight experiences about six price adjustments within 90 days. In terms of bookings, the average daily booking rate is less than one. Roughly 40% of observed bookings are for local traffic, the remaining are flow bookings. At the departure time, average load factors are 72%, which is lower than the industry average of about 80% for this time period. We do observe sellouts for all competitors in the data.

In Figure 7 we plot average fares and booking rates by day before departure. The left

Table 1: Summary statistics

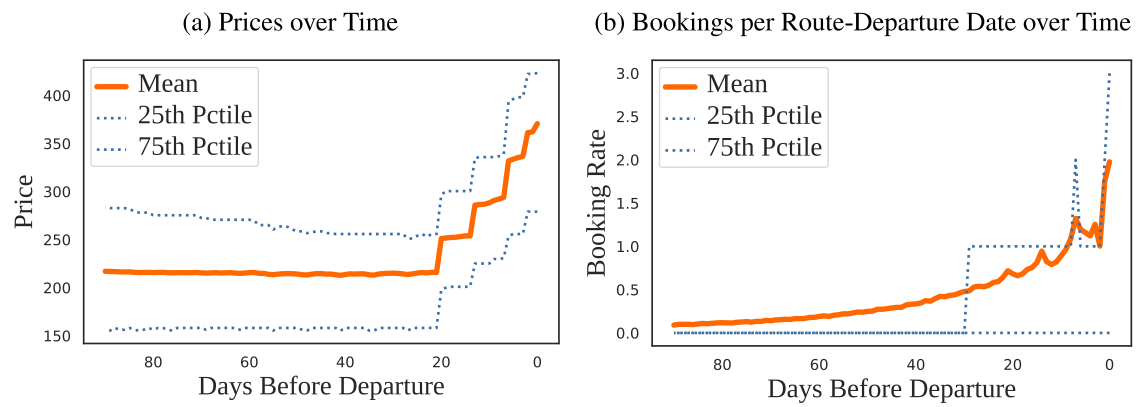
Data Series	Variable	Mean	Std. Dev.	Median	5th pctile	95th pctile
Fares						
	One-Way Fare ($)	233.7	111.4	218.6	92.1	390.7
	Num. Fare Changes	6.4	2.4	6.0	3.0	11.0
Bookings						
	Booking Rate-local	0.2	0.6	0.0	0.0	1.0
	Booking Rate-all	0.5	1.2	0.0	0.0	3.0
	Ending LF (%)	72.1	19.8	76.0	32.9	98.0

Note: One-Way fare is for the lowest economy class ticket available for purchase. Number of fare changes records the number of price adjustments observed for each flight. Booking rate-local excludes flow traffic. Booking rate-all includes both local and flow traffic. Ending load factor (LF) reports the percentage of seats occupied at departure time.

panel (prices) shows that average fares are fairly flat between 90 and 21 days before departure. The top end of the distribution is decreasing in this time window. There are noticeable "steps" in the last 21 days before departure which highlights the use of advance purchase (AP) discounts in the industry. In the routes examined, we observe AP requirements at 21, 14, 7, and 3 days before departure. In the right panel (bookings) we highlight that bookings increase as the departure date approaches. This coincides with increasing prices and suggests that demand becomes more inelastic over time. The booking rate is greater than one per flight over the last month before departure.

In Figure 8 we focus on outcomes across competitors. The left panel provides a scatter plot of ending load factor at the route-departure date level for the entire data sample. The orange squares present route-level load factors. Note there exists a large mass of points both above and below the 45-degree line—one competitor does not consistently sell a larger fraction of capacity than the other carrier in all markets. We do observe some flights with substantial overselling. In our analysis, we restrict firms to selling at most their capacity. In the right panel we plot the average fare difference across competitors over time when exactly two flights are offered. Note that fares tend to be similar across competitors—the average difference is less than $10. However, the gradient of the prices differs. One competitor has relatively higher prices well in advance of departure and relatively lower

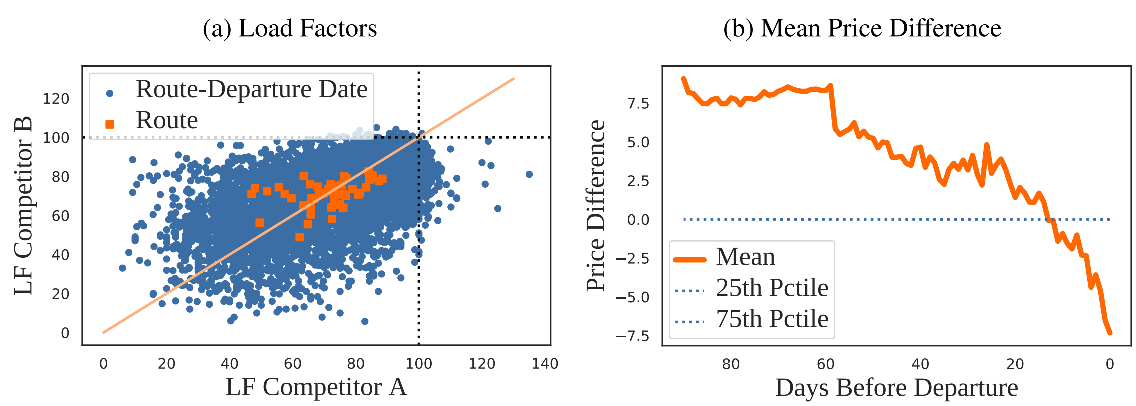
Figure 7: Prices and Bookings by Day Before Departure



Note: Panel (a) shows the average and interquartile range of flight prices over time. Panel (b) shows the average and interquartile range of flight booking rates over time. Greater than 30 days before departure, the 25th and 75th percentiles coincide.

prices close to departure. Note that for over 50% of the data, prices across firms are equal, that is, there is substantial price matching.

Figure 8: Load Factor and Price Differences across Carriers



Note: Panel (a) shows the average load factor (across all flights) at the route-departure date level for both competitors in blue. The orange squares report average route-level load factors. The diagonal line is the 45-degree line. Panel (b) shows the average and the 25th and 75 percentiles of the difference in prices for markets in which exactly two flights across firms are offered (one flight per airline).

5 Demand Model and Estimates

5.1 Empirical Specification

We model nonstop air travel demand using a nested logit demand model. Our model differs from recent empirical work on airlines that use a mixed-logit model to model "business" and "leisure" travelers (Lazarev, 2013; Williams, 2022; Aryal et al., 2021; Hortaçsu et al., 2021). We use a flexible nested-logit model with time-varying as it better maps to our theoretical model and results in unique equilibrium price paths.[8]

Define a market as an origin-destination (r), departure date (d), and day before departure (t) combination. Each flight j, leaving on date d, is modeled across $t \in \{0, ..., T\}$. The first period of sale is $t = 0$, and the flight departs at T. We use a 90-day time horizon. With daily data, we model demand at the daily level. Arriving consumers choose flights from the choice set $J_{t,d,r}$ that maximize their individual utilities, or select the outside option, $j = 0$. There are two nests. The outside good belongs to its own nest, and all inside goods belong to the second nest.

We specify consumer arrivals to be

$$\lambda_{t,d,r} = \exp\left(\tau_r^{\text{OD}} + \tau_d^{\text{DD}} + \tau_{t,d}^{\text{SD}} + f(\text{DFD})_t\right),$$

where τ denote fixed effects for the route, departure date, and search date; $f(\cdot)$ is a polynomial series of degree three. We scale up these estimated arrival rates using hyperparameters to account for unobserved searches.

Conditional on arrival, we specify consumer utilities as

$$u_{i,j,t,d,r} = \mathbf{x}_{j,t,d,r}\boldsymbol{\beta} - \alpha_t p_{j,t,d,r} + \zeta_{i,J} + (1-\sigma)\varepsilon_{i,j,t,d,r},$$

where $\zeta_{i,J} + (1-\sigma)\varepsilon_{i,j,t,d,r}$ follows a type-1 extreme value distribution, and $\zeta_{i,J}$ is an idiosyncratic preference for the inside goods. The parameter $\sigma \in [0,1]$ denotes correlation

[8]The mass-point random coefficients models yields multiple equilibria in our setting.

in preferences within the nests. We allow the price sensitivity parameter to vary over time (α_t) using three-day intervals of time; hence, we estimate 30 price sensitivity parameters. We include a number of covariates in $\mathbf{x}$ where preferences are assumed to not vary across t: departure week of the year, departure day of the week, route, carrier, and departure time fixed effects.

Each arriving consumer solves their utility maximization problem such that consumer i chooses flight j if, and only if,

$$u_{i,j,t,d,r} \geq u_{i,j',d,t,r}, \forall j' \in \mathscr{J}_{t,d,r} \cup \{0\}.$$

Temporarily dropping the t, d, r subscripts, we define

$$D_J := \sum_{j \in J} \exp\left(\frac{\mathbf{x}_j \boldsymbol{\beta} - \alpha_t p_j}{1-\sigma}\right),$$

so that the probability that a consumer purchases j within the set of inside goods is equal to

$$s_{j|J} = \frac{\exp\left(\frac{\mathbf{x}_j \boldsymbol{\beta}_t - \alpha_t p_j}{1-\sigma}\right)}{D_J}.$$

It follows that the probability that a consumer purchases any inside good product is equal to

$$s_J = \frac{D_J^{1-\sigma}}{1 + D_J^{1-\sigma}}.$$

We define overall product shares to be equal to $s_j = s_{j|J} \cdot s_J$, which are implicitly at the market level (t, d, r).

Our assumptions imply that demand is distributed Poisson with a product purchase rate equal to $\min\left\{\lambda_{t,d,r} \cdot s_{j,t,d,r}, C_{j,t,d,r}\right\}$. Note as the length of a period decreases, at most one seat will be sold in any period.

5.2 Demand Estimates

We estimate the model in two steps. In the first step, we estimate the arrival process parameters using Poisson regressions. We then estimate preferences of the Poisson demand model using maximum likelihood. We estimate standard errors using bootstrap.

We follow Hortaçsu et al. (2021) in constructing arrivals using clickstream data for one airline. These data track all "clicks" or interactions on the firm's websites. We first sum the number of searches corresponding to each market (r, d, t) and then we scale up estimated arrival rates to account for unobserved searches. This follows from a property of the Poisson distribution and the assumption that consumers who search/purchase through alternative platforms (travel agents, other carriers' websites) have the same underlying preferences. We first calculate the fraction of direct bookings by day before departure and then scale up the estimated arrival rates using these these fractions. This adjusts arrivals for a single carrier. In our preferred specification, we then double these arrival rates to account for competitor indirect and direct searches, both of which are unobserved to us. We conduct robustness to this hyperparameter in Appendix D.

We summarize the demand estimates in Table 2. We estimate the nesting parameter to be 0.5 so that there is substantial substitution within inside goods. The price sensitivity parameters vary by nearly a factor of ten over time. We present a time series plot of α_t in Figure 9. Almost all of our controls are significant, with day of the week and week of the year having the most influence on market shares. The competitor FE is significantly less important in driving variation in shares. We estimate the average own-price elasticity to be -1.4.

In Figure 9-(a), we plot average adjusted arrival rates as well as parts of the distribution (5%, 25%, 75%, 95%) across markets. We estimate just a few arrivals per market 90 days before departure that then increases to over 10 passengers per day close to departure. Recall that the average booking rate across flights is less than 2.0 (see Figure 7) so that market shares are low. An increase in interest in travel is a general findings across all of the routes in our sample. Note that while the 75th percentile closely followed the mean, the

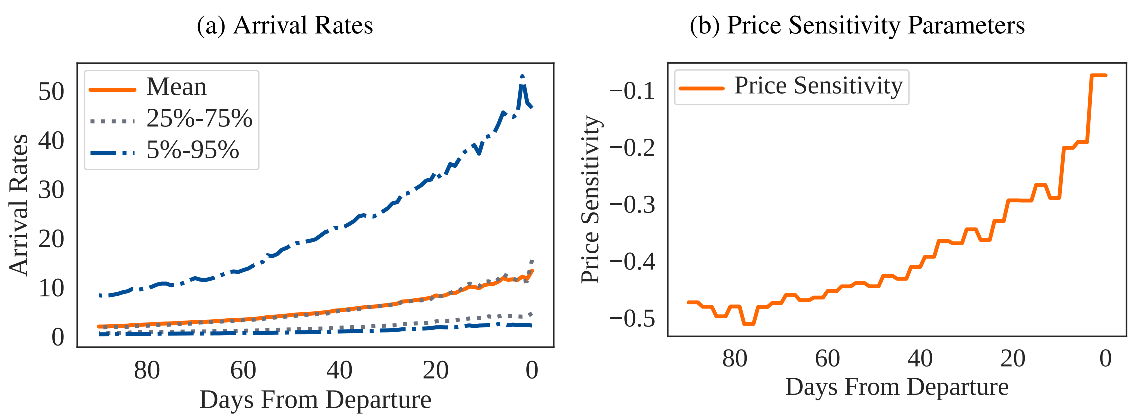
Table 2: Demand Estimates Summary Table

Variable	Symbol	Estimate	Std. Error.	Range	% Sig.
Nesting Parameter	σ	0.498	0.010	–	–
Price Sens.	α	–	–	[-0.511 ,-0.074]	100.0
Competitor FE	–	0.071	0.003	–	–
Day of Week FE	–	–	–	[-1.637 ,-0.961]	100.0
Departure Time FE	–	–	–	[-0.462 ,-0.050]	100.0
Route FE	–	–	–	[-0.177 ,0.226]	94.4
Week FE	–	–	–	[-0.953 ,0.699]	86.0
Sample Size	N		2,814,686		
Avg Elasticity	e^D		-1.438		

Note: Demand estimates for the 58 routes in our sample.

top part of the distribution is substantially higher, which corresponds to the routes in the upper-right of Figure 6-(d).

Figure 9: Arrival Rates and the Price Sensitivity Parameters



Note: Panel (a) shows fixed values, adjusted for unobserved searches, of arrival rates over time. The mean is the average arrival rate across all markets. The percentiles are also over markets. Panel (b) shows our estimates of the price sensitivity parameters in 3-day groupings.

6 Counterfactual Analysis

With our demand estimates, we quantify the welfare effects of dynamic price competition using three sets of counterfactuals—the benchmark, lagged, and deterministic models presented in our theoretical analysis.

Although the benchmark model holds for an arbitrary number of firms and products, computing equilibria of the game is difficult. We adjust our empirical estimates in a number of ways for computational reasons:

i) We consider only two products. Instead of investigating pricing in routes where we observe a single flight operating by each firm, we adjust the choice set, utilities, and capacities for all routes.

ii) We take the mean utilities across observed flights for each departure date and an input.

iii) We take the maximum observed capacity for each route-carrier-departure date. Although it may be natural to sum the capacities when restricting the choice set, we have found that large capacities presents a significant computational burden.

iv) We use the observed arrival process for each route-departure date. We do not adjust the estimated arrival processes as the inside good shares tend to be small. That is, because most consumers choose the outside good, we do not scale down arrival rates to account for smaller choice sets.

v) Finally, we handle flow (connecting traffic) bookings two ways. In our reported counterfactuals here, we model these bookings via Poisson processes that the firm does not internalize when pricing local demand. In the appendix we report counterfactuals where we subtract off all connecting bookings at the start of the game. This affects market outcomes because it reduces uncertainty for firms.

Benchmark Model

We approximate the continuous time model to solve for equilibrium prices for every departure date. We consider hourly decisions over 90 days. Both firms start with initial capacities C_f and $C_{f'}$. We solve via backward induction, which we outline here. In the last pricing period, $t = T$, both $\Pi_T(\mathbf{K}) = 0$ and $\Omega_T(\mathbf{K}) = \mathbf{0}$. Therefore, both firms solve static revenue maximize problems. We set the best response functions equal to zero and solve for the fixed point. Label this $p_T = EQ_T(\Omega_T = \mathbf{0})$. Using the differential equation $\dot{\Pi}_T(\mathbf{K}) = -\lambda_T \pi_T(p_T(\mathbf{K}), \Omega_T = \mathbf{0})$, where $\pi_T(p_T(\mathbf{K}), \Omega_T = \mathbf{0})$ are the stage-game payoffs, we can calculate $\Pi_{T-\Delta}(\mathbf{K}) = -\Delta \cdot \dot{\Pi}_T(\mathbf{K})$ and $\Omega^f_{f',T-\Delta}(\mathbf{K}) = \Pi_{f,T-\Delta}(\mathbf{K}) - \Pi_{f,T-\Delta}(\mathbf{K} - \mathbf{e}_{f'})$. Given the updated own and scarcity effect parameters we again solve for equilibrium prices, $p_{T-\Delta} = EQ_{T-\Delta}(\Omega_{T-\Delta})$.[9] We continue backwards in time to $t = 0$.

Due to the large number of state variables present in many of our routes, we store Ω_t and p_t every 24 hours (at the start of a day) to use in counterfactual simulations. We will then appeal to modeling demand via multinomial distributions after drawing arrivals via Poisson distributions in lieu of studying each consumer's individual choice after drawing arrivals via Bernoulli distributions.

Lagged-Price and Deterministic Models

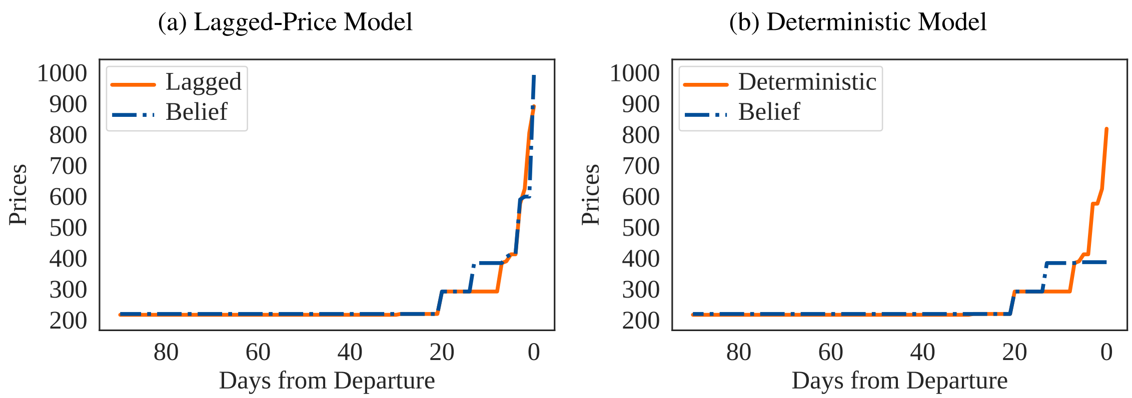
Recall that both the lagged-price and deterministic models use discrete fares. All airlines use discrete fares, and our data allow us to create fare menus for all carrier, route combinations.[10] More specifically, airlines file fares for "buckets." Typically, each carrier fills between seven and fifteen buckets per route. Buckets can change by day before departure, i.e., the fare for a given bucket increases. However, the data suggests that a more consequential change in buckets over time is their availability. Oftentimes, a fare is restricted for a certain time period before departure—an advance purchase discount. For example, Figure 10-(a) shows an example fare menu for a given carrier-route in the data. Prices vary

[9] We use a modified Powell method from MINPACK's hybrid routine to solve the best=response functions equal to each other.

[10] See Hortaçsu et al. (2021) for more details.

from less than \$200 to over \$3,000. In Figure 10-(b) and Figure 10-(c), we provide example price paths for the lagged and deterministic models using our empirical estimates.

Figure 10: Heuristic Models Pricing Example



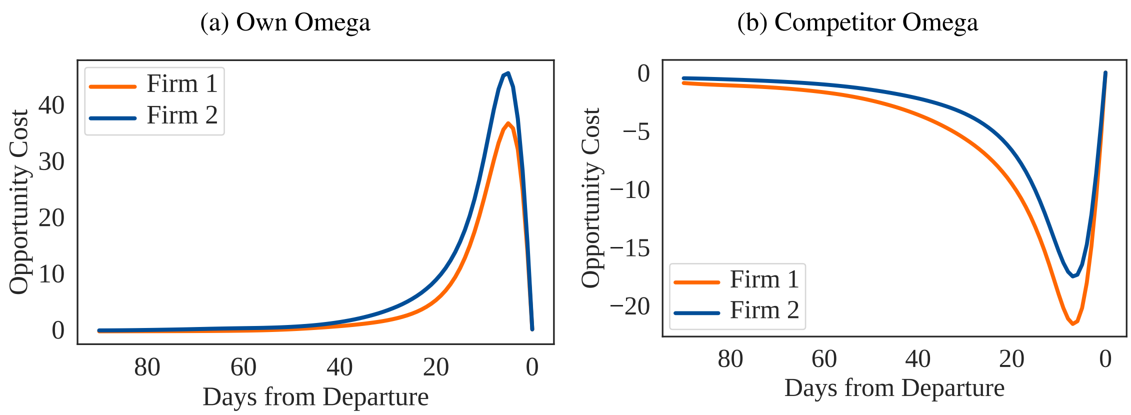
Note:

Implementation

We conduct 10,000 Monte Carlo experiments for every route, departure date combination. We simulate all counterfactuals twice, one where flow traffic is subtracted from initial observed capacity is advance, and one where flow traffic is modeled through Poisson processes, not internalized when pricing local demand. We store prices, arrivals, quantities sold, and calculate consumer surplus and revenues for every market.

6.1 Welfare Effects of Dynamic Price Competition

Figure 11: Benchmark Model Opportunity Costs



Note: Panel (a) reports the own-firm opportunity cost over time for both firms. Panel (b) reports the cross-firm competitor opportunity cost over time for both firms.

Table 3: Counterfactual Results

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
– own ω only	99.4	99.5	99.6	100.5	100.1	100.4	100.1	101.1
– no ω	96.5	96.2	95.9	101.3	99.2	101.1	100.1	102.6
Deterministic	98.3	96.8	97.6	108.4	103.9	103.2	101.2	109.9
Lagged	105.2	101.7	102.7	103.9	103.2	99.6	99.9	98.8
Uniform	118.2	85.7	87.4	112.9	102.2	93.6	97.5	72.6

Note:

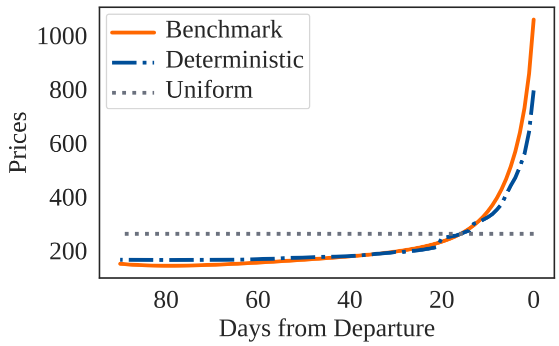
Table 4: Firm Profits Across Counterfactuals

Firm 1 Preference	Firm 2 Preference	Fraction of Markets (r, d)
Benchmark	Benchmark	57.0%
Deterministic	Deterministic	23.1%
Deterministic	Benchmark	6.2%
Benchmark	Deterministic	13.8%

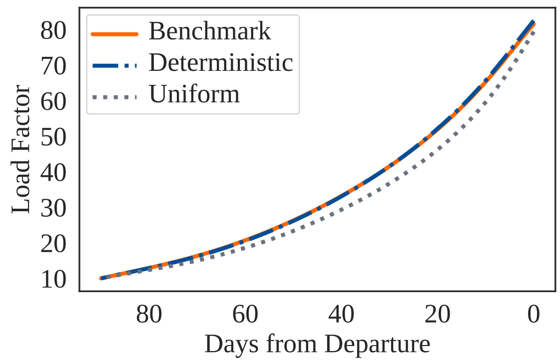
Note:

Figure 12: Counterfactual Summary Plots

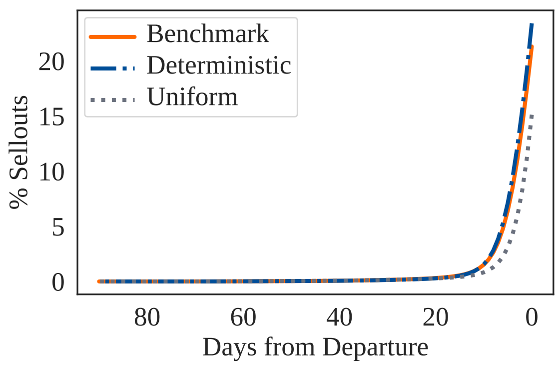
(a) Prices over Time



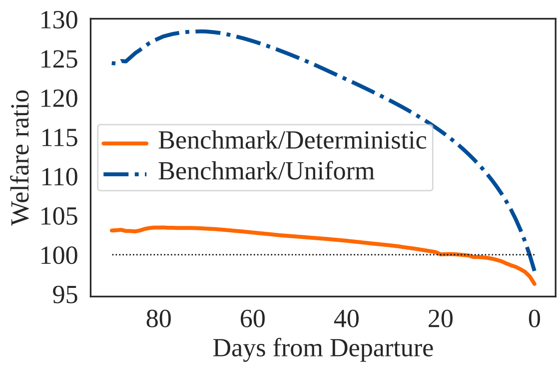
(b) Load Factors over Time



(c) Sellouts over Time



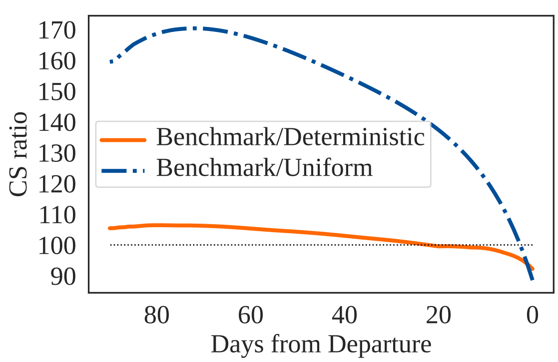
(d) Cumulative Welfare Comparison



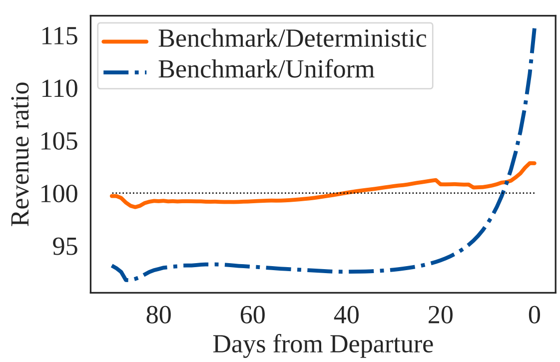
Note:

Figure 13: Cumulative Surplus Differences Across Counterfatuals

(a) Cumulative CS Difference



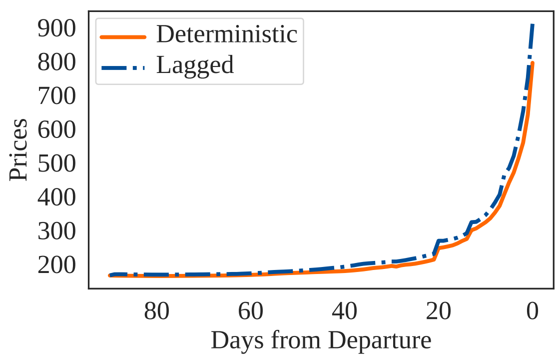
(b) Cumulative Revenue Difference



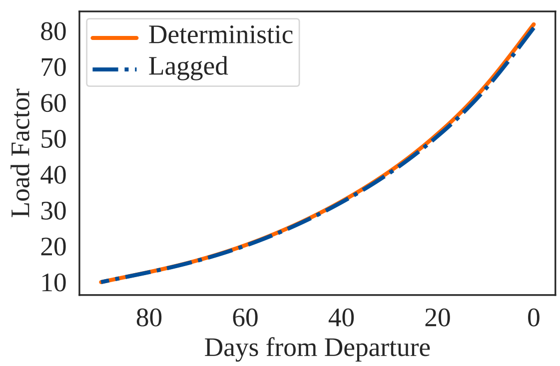
Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Figure 14: Heuristic Counterfactuals Summary Plots

(a) Prices over Time



(b) Load Factors over Time



Note:

References

Aryal, Gaurab, Charles Murry, and Jonathan W Williams, "Price discrimination in international airline markets," *arXiv preprint arXiv:2102.05751*, 2021.

Asker, John, Chaim Fershtman, and Ariel Pakes, "Artificial Intelligence and Pricing: The Impact of Algorithm Design," Technical Report, National Bureau of Economic Research 2021.

Bergemann, Dirk and Juuso Välimäki, "Dynamic price competition," *Journal of Economic Theory*, 2006, *127* (1), 232–263.

Berry, Steven and Panle Jia, "Tracing the woes: An empirical analysis of the airline industry," *American Economic Journal: Microeconomics*, 2010, *2* (3), 1–43.

Board, Simon and Andrzej Skrzypacz, "Revenue management with forward-looking buyers," *Journal of Political Economy*, 2016, *124* (4), 1046–1087.

Borenstein, Severin and Nancy L Rose, "Competition and price dispersion in the US airline industry," *Journal of Political Economy*, 1994, *102* (4), 653–683.

Burgess, CE, "Continuous functions and connected graphs," *The American Mathematical Monthly*, 1990, *97* (4), 337–339.

Calvano, Emilio, Giacomo Calzolari, Vincenzo Denicolo, and Sergio Pastorello, "Artificial intelligence, algorithmic pricing, and collusion," *American Economic Review*, 2020, *110* (10), 3267–97.

Caplin, Andrew and Barry Nalebuff, "Aggregation and imperfect competition: On the existence of equilibrium," *Econometrica: Journal of the Econometric Society*, 1991, pp. 25–59.

Dana, James D, "Equilibrium price dispersion under demand uncertainty: the roles of costly capacity and market structure," *The RAND Journal of Economics*, 1999, pp. 632–660.

_ **and Kevin R Williams**, "Intertemporal price discrimination in sequential quantity-price games," *Marketing Science*, 2022.

de Albéniz, Victor Martínez and Kalyan Talluri, "Dynamic price competition with fixed capacities," *Management Science*, 2011, *57* (6), 1078–1093.

Dilme, Francesc and Fei Li, "Revenue management without commitment: Dynamic pricing and periodic flash sales," *The Review of Economic Studies*, 2019, *86* (5), 1999–2034.

Dontchev, Asen L and R Tyrrell Rockafellar, *Implicit functions and solution mappings*, Vol. 543, Springer, 2009.

Dudey, Marc, "Dynamic edgeworth-bertrand competition," *The Quarterly Journal of Economics*, 1992, *107* (4), 1461–1477.

Fleming, Wendell H and Halil Mete Soner, *Controlled Markov processes and viscosity solutions*, Vol. 25, Springer Science & Business Media, 2006.

Gallego, Guillermo and Garrett Van Ryzin, "Optimal dynamic pricing of inventories with stochastic demand over finite horizons," *Management science*, 1994, *40* (8), 999–1020.

Gerardi, Kristopher S and Adam Hale Shapiro, "Does competition reduce price dispersion? New evidence from the airline industry," *Journal of Political Economy*, 2009, *117* (1), 1–37.

Gershkov, Alex, Benny Moldovanu, and Philipp Strack, "Revenue-maximizing mechanisms with strategic customers and unknown, markovian demand," *Management Science*, 2018, *64* (5), 2031–2046.

Hansen, Karsten T, Kanishka Misra, and Mallesh M Pai, "Frontiers: Algorithmic collusion: Supra-competitive prices via independent algorithms," *Marketing Science*, 2021, *40* (1), 1–12.

Horn, Roger A and Charles R Johnson, *Matrix analysis*, Cambridge university press, 2012.

Hortaçsu, Ali, Olivia R Natan, Hayden Parsley, Timothy Schwieg, and Kevin R Williams, "Organizational structure and pricing: Evidence from a large us airline," Technical Report, National Bureau of Economic Research 2021.

Kellogg, RB, "Uniqueness in the Schauder fixed point theorem," *Proceedings of the American Mathematical Society*, 1976, *60* (1), 207–210.

Konovalov, Alexander and Zsolt Sándor, "On price equilibrium with multi-product firms," *Economic Theory*, 2010, *44* (2), 271–292.

Lazarev, John, "The welfare effects of intertemporal price discrimination: an empirical analysis of airline pricing in US monopoly markets," 2013.

Leisten, Matthew, "Algorithmic Competition, with Humans," 2021.

Maskin, Eric and Jean Tirole, "A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles," *Econometrica: Journal of the Econometric Society*, 1988, pp. 571–599.

McAfee, R Preston and Vera Te Velde, "Dynamic pricing in the airline industry," *Handbook on economics and information systems*, 2006, *1*, 527–67.

Milgrom, Paul and John Roberts, "Rationalizability, learning, and equilibrium in games with strategic complementarities," *Econometrica: Journal of the Econometric Society*, 1990, pp. 1255–1277.

Nocke, Volker and Nicolas Schutz, "Multiproduct-firm oligopoly: An aggregative games approach," *Econometrica*, 2018, *86* (2), 523–557.

Puller, Steven L, Anirban Sengupta, and Steven N Wiggins, "Does scarcity drive intra-route price dispersion in airlines?"

Sengupta, Anirban and Steven N Wiggins, "Airline pricing, price dispersion, and ticket characteristics on and off the internet," *American Economic Journal: Economic Policy*, 2014, *6* (1), 272–307.

Siegert, Caspar and Robert Ulbricht, "Dynamic oligopoly pricing: Evidence from the airline industry," *International Journal of Industrial Organization*, 2020, *71*, 102639.

Solan, Eilon and Omri N Solan, "Browder's Theorem through Brouwer's Fixed Point Theorem," *arXiv preprint arXiv:2107.02428*, 2021.

Stavins, Joanna, "Price discrimination in the airline market: The effect of market concentration," *Review of Economics and Statistics*, 2001, *83* (1), 200–202.

Sweeting, Andrew, James W Roberts, and Chris Gedge, "A model of dynamic limit pricing with an application to the airline industry," *Journal of Political Economy*, 2020, *128* (3), 1148–1193.

Talluri, Kalyan and Garrett Van Ryzin, "Revenue management under a general discrete choice model of consumer behavior," *Management Science*, 2004, *50* (1), 15–33.

Williams, Kevin R., "The Welfare Effects of Dynamic Pricing: Evidence From Airline Markets," *Econometrica*, 2022, *90* (2), 831–858.

Zhao, Wen and Yu-Sheng Zheng, "Optimal dynamic pricing for perishable assets with nonhomogeneous demand," *Management science*, 2000, *46* (3), 375–388.

A General Model with Perfect Information

[Appendix A is work in progress]

A.1 Model Setup

We consider a market with $F \geq 1$ firms and $J \geq F$ products. Denote the set of firms by $\mathscr{F} := \{1, \ldots, F\}$ and the set of products by $\mathscr{J} := \{1, \ldots, J\}$. Each firm $f \in \mathscr{F}$ sells products in the set $\mathscr{J}_f$, where $(\mathscr{J}_f)_{f \in \mathscr{F}}$ is a partition of $\mathscr{J}$; that is, $\mathscr{J} = \bigcup_{f \in \mathscr{F}} \mathscr{J}_f$ and $\mathscr{J}_f \cap \mathscr{J}_{f'} = \emptyset$ for $f \neq f'$. Thus, no product is sold by more than one firm. Each firm f is equipped with an initial inventory of their products $j \in \mathscr{J}_f$, denoted by $K_0^j \in \mathbb{N}$. We abstract from the initial capacity choice and do not model capacity costs. The costs modeled are the costs due to scarcity — the opportunity costs of each seat.

Products are imperfect substitutes and must be scrapped with zero value at a deadline $T \in \mathbb{R}$. We analyze a discrete-time game with periods $t \in \{0, \Delta, \ldots, T - \Delta\}$, $\Delta > 0$, and then consider the dynamics for the continuous time approximation as $\Delta \to 0$.

In every period t, each firm f simultaneously chooses prices $p_{j,t}$ for its products $j \in \mathscr{J}_f$ for the time interval $[t, t + \Delta)$. Firms and consumers observe the entire history of prices and capacities. If a product j is sold, the firm f with $j \in \mathscr{J}_f$ receives a payoff of p_j. Each firm's total profit is simply the sum of payoffs in all periods.

In this game, the payoff-relevant state is given by the vector of inventories at time t $\mathbf{K}_t = (K_{j,t})_{j \in \mathscr{J}}$ and the time t. We are interested in Markov perfect equilibria in which each firm's strategy is measurable with respect to $(\mathbf{K}, t)$. We denote a Markov pricing strategy of firm f by $\mathbf{p}_t^f(\mathbf{K}) = (p_{j,t}(\mathbf{K}))_{j \in \mathscr{J}_f}$.

For any generic vector $\mathbf{x} = (x_j)_{j \in \mathscr{J}}$, we denote by $\mathbf{x}_{-j}$ the vector $(x_{j'})_{j' \neq j}$ that excludes dimension j. We let $\mathbf{e}_j = (0, ..., 1, ...0)$ denote the unit vector with a 1 in dimension j and zeros in all other entries, and $\mathbf{1} = (1, \ldots, 1)$ the vector of ones.

Discrete choice and logit demand. A commonly used demand specification is a discrete choice logit model where consumers are assumed to be short-lived. In the classic logit model, a consumer t who purchases a product j and pays price $p_{j,t}$ experiences a utility of

$$u_{j,t} = \delta_j - \alpha_t p_{j,t} + \varepsilon_{j,t}$$

where $\varepsilon_{j,t} \in \mathbb{R}$ are independently distributed, e.g., according to a type 1 extreme value (T1EV) distribution with mean zero in the commonly used logit environment. The parameters $\delta_j \in \mathbb{R}$ are the average mean utility for each product j, and α_t is the marginal utility to income. We assume that α_t is continuous in t, so the demand parameters are given by $\boldsymbol{\theta}_t = ((\delta_j)_{j\in\mathscr{J}}, \alpha_t)$. Finally, we assume that the utility from the outside option is given by $u_{0,t} = \epsilon_{0,t}$, where $\epsilon_{0,t}$ are independent of the other idiosyncratic terms and also distributed according to T1EV. Then, the utility maximization problem of the short-lived consumer is simply to choose a $j \in \mathscr{J}_0 := \mathscr{J} \cup \{0\}$ that maximizes her individual utility and we can write

$$s_j(\mathbf{p}; (\delta_j)_{j\in\mathscr{J}}, \alpha_t) = \frac{e^{\delta_j - \alpha_t p_j}}{1 + \sum_{j\in\mathscr{J}} e^{\delta_j - \alpha_t p_j}}.$$

We offer a detailed analysis of this specification in Appendix C. In particular, we show that it satisfies Assumption 1-5 below.

A.2 The Oligopoly Case

Given a vector of inventory $\mathbf{K}$, we denote the value function of firm f at time t in an equilibrium (if one exists) by $\Pi_t^f(\mathbf{K};\Delta)$ and the equilibrium price vector by $\mathbf{p}_t^*(\mathbf{K};\Delta)$. Then:

$$\begin{aligned}\Pi_t^f(\mathbf{K};\Delta) = \quad & \Delta\lambda_t \Bigg(\underbrace{\sum_{j\in\mathscr{J}_f} s_{j,t}(\mathbf{p}_t^*(\mathbf{K};\Delta))\,(p_{j,t}^*(\mathbf{K};\Delta)+\Pi_{t+\Delta}^f(\mathbf{K}-\mathbf{e}_j;\Delta))+}_{\text{revenue of own sale}} \\ & \underbrace{\sum_{j'\neq\mathscr{J}\setminus\mathscr{J}_f} s_{j',t}(\mathbf{p}_t^*(\mathbf{K};\Delta))\,\Pi_{t+\Delta}^f(\mathbf{K}-\mathbf{e}_{j'};\Delta)}_{\text{continuation value if } j' \text{ is sold}}\Bigg) + \underbrace{\Bigg(1-\Delta\lambda_t\sum_{j'\in\mathscr{J}} s_{j'}(\mathbf{p}_t^*(\mathbf{K};\Delta))\Bigg)}_{\text{probability of no purchase}}\Pi_{t+\Delta}^f(\mathbf{K};\Delta),\end{aligned}$$

with boundary conditions $\Pi_t^f(\mathbf{K};\Delta)\equiv 0$ with $K_j=0$ for all $j\in\mathscr{J}_f$ and $\Pi_T^f(\mathbf{K})\equiv 0$ for all $\mathbf{K}$. Then, we denote the *scarcity effect of firm f for product j* in state $(\mathbf{K},t)$ by

$$\omega_{j,t}^f(\mathbf{K};\Delta) := \Pi_{t+\Delta}^f(\mathbf{K};\Delta) - \Pi_{t+\Delta}^f(\mathbf{K}-\mathbf{e}_j;\Delta)$$

where we set $\omega_{j,t}^f(\mathbf{K};\Delta):=\infty$ if $K_j=0$ for $j\in\mathscr{J}_f$, and $\omega_{j,t}^f(\mathbf{K};\Delta):=0$ if $K_j=0$ for $j\notin\mathscr{J}_f$. We denote the matrix of opportunity costs by

$$\Omega_t(\mathbf{K};\Delta) = (\omega_{j,t}^f(\mathbf{K};\Delta))_{f,j} \in \mathbb{R}^{\mathscr{F}}\times\mathbb{R}^{\mathscr{J}^0},$$

where $\mathscr{J}^0=\mathscr{J}\cup\{0\}$ and $\omega_{0,t}^f(\mathbf{K};\Delta)=0$.

In any Markov equilibrium, the equilibrium prices $p_{j,t}^*(\mathbf{K})$ must be a Nash equilibrium of a stage game in which each firm f simultaneously chooses prices $\mathbf{p}^f=(p_j)_{j\in\mathscr{J}_f}$ and where given a price vector $\mathbf{p}$ each firm f receives a payoff of

$$\sum_{j\in\mathscr{J}_f} s_{j,t}(\mathbf{p})(p_j-\omega_{j,t}^f(\mathbf{K};\Delta)) - \sum_{j'\notin\mathscr{J}_f} s_{j',t}(\mathbf{p})\omega_{j',t}^f(\mathbf{K};\Delta).$$

Note that the payoffs of firm f depend on the demand for prodcts of all other firms, so that we cannot apply results from Caplin and Nalebuff (1991) and the payoffs are also neither super-modular nor log-supermodular, so we cannot apply results from Milgrom and Roberts (1990). The game is also not a potential game. The best response functions of each firm are closest to the ones discussed in Nocke and Schutz (2018).

A.2.1 The stage game

Consider a stage game in which for a given cost matrix $\Omega \in \mathbb{R}^{\mathscr{F}} \times \mathbb{R}^{\mathscr{J}^0}$, firms simultaneously set prices where their payoffs are given by

$$\sum_{j \in \mathscr{J}_f} s_j(\mathbf{p})(p_j - \omega_j^f) - \sum_{j' \notin \mathscr{J}_f} s_{j'}(\mathbf{p})\omega_{j'}^f.$$

Then, in order to guarantee that the best responses of firms are bounded, we assume:

General Assumption 3 (Boundedness of cross derivatives)**.** *For any* $j, j' \in \mathscr{J}$, $j \neq j'$, $\frac{1}{J-1}\left|\frac{\partial s_j}{\partial p_j}(\mathbf{p})\right| > \left|\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p})\right|$ *for all price vectors* $\mathbf{p}$.

We show in the proof of Lemma 2 that this assumption, together with Assumption **??**, guarantees that a firm f's best-response prices $\mathbf{p}^f$ must satisfy

$$\mathbf{p}^f = \left(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p})\right)^{-1} D_{\mathbf{p}^f}\left(\mathbf{s}(\mathbf{p})^\intercal \boldsymbol{\omega}^f\right) - \left(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p})\right)^{-1}\mathbf{s}^f(\mathbf{p}) =: g^f(\mathbf{p}). \tag{9}$$

Then, by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010), $g(\mathbf{p}) := (g^f(\mathbf{p}))_f$ has a unique fixed point if and only if the following assumption is satisfied.

General Assumption 4. $\det\left(D_{\mathbf{p}}\big(g(\mathbf{p})\big) - I\right) \neq 0$ *for all* $\mathbf{p}$.

General Lemma 2. *Let Assumptions 1,2, 3, and 4 be satisfied. Then, the stage game admits a unique equilibrium.*

Given Assumptions **??**, 4 is satisfied for any matrix of opportunity costs Ω in a neighborhood $\mathscr{O}$ that contains the zero matrix $\Omega = \mathbf{0}$ (by continuity). Consequently, close to the deadline T when $\Omega = \mathbf{0}$, the stage games are well behaved. However, further away from the deadline, when opportunity costs can potentially become large, the stage game may be less well-behaved as we show in the main text.

For the demand specification that satisfy the commonly used assumption of "Independence of Irrelevant Alternatives (IIA)" we can, however, establish existence more generally and also derive an economically meaningful markup formula. First, note that the IIA assumption states the following:

General Assumption 5 (Independence of Irrelevant Alternatives (IIA))**.** $\frac{\partial}{\partial p_j}\frac{s_{j_1}(\mathbf{p})}{s_{j_2}(\mathbf{p})} = 0$ *for* $j \neq j_1, j_2$.

Given Assumptions 1-3, we can show that the game with multi-product firms can be transformed into a game of single-product firms.

General Proposition 2 (Mark-up formula under IIA)**.** *Under Assumptions 1-**??**, and 5, there exist functions* $d_j(\mathbf{p}_{-f};\boldsymbol{\omega},\boldsymbol{\theta})$ *so that the equilibrium prices of the stage game coincide with the equilibrium prices of a game with a set* $\mathcal{J}$ *of players who each simultaneously choose a price* p_j *maximizing*

$$s_j(\mathbf{p})(p_j - c_j(\mathbf{p}_{-j};\boldsymbol{\omega},\boldsymbol{\theta})) + d_j(\mathbf{p}_{-j};\boldsymbol{\omega},\boldsymbol{\theta})$$

with a cost function

$$c_j(\mathbf{p}_{-j};\boldsymbol{\omega},\boldsymbol{\theta}) := \omega_j^f - \sum_{j'\in\mathcal{J}_f\setminus\{j\}} \tilde{s}_{j,j'}(\mathbf{p}_{-j})(p_{j'} - \omega_j^f) + \sum_{j'\notin\mathcal{J}_f} \tilde{s}_{j,j'}(\mathbf{p}_{-j})\omega_{j'}^f \tag{10}$$

and $\tilde{s}_{j,j'}(\mathbf{p}_{-j}) := \frac{\frac{\partial s_{j'}}{\partial p_j}(\mathbf{p})}{\sum_{j''\in\mathcal{J}^0\setminus\{j\}} \frac{\partial s_{j''}}{\partial p_j}(\mathbf{p})}$.

A consequence of Proposition 3 is that the first-order conditions (FOCs) that implicitly define the best response functions of the firms, can be written in a markup formulation as

$$\frac{p_j - c_j(\mathbf{p}_{-j};\boldsymbol{\omega},\boldsymbol{\theta})}{p_j} = -\frac{1}{\epsilon_j(\mathbf{p})} \tag{11}$$

where $\epsilon_j(\mathbf{p}) = \frac{\partial s_j(\mathbf{p})}{\partial p_j}\frac{p_j}{s_j(\mathbf{p})}$ is the elasticity of demand. The formulation (11) emphasizes the impact of the competitive forces in the presence of opportunity costs: Other firm's prices do not only impact own demand, but also the effective cost of selling the product.

Lemma 7 (Existence)**.** *Assume that Assumptions 1-**??**, and 5 are satisfied. Then, there exists an equilibrium to the above stage game for any cost matrix* Ω*.*

Lemma 8. *If the equilibrium of the stage game is unique for a compact set* $\mathcal{O}$ *of costs* Ω*, then there exists an equilibrium price vector* $\mathbf{p}^*(\Omega, \boldsymbol{\theta})$ *that is continuous in* Ω *on* $\mathcal{O}$ *and* $\boldsymbol{\theta}$ *on* Θ*.*

A.2.2 The continuous-time limit

Lemma 9 (Continuous-time limit Limit)**.** *Let us assume that Assumptions **??**-4 are satisfied for all stage games. Then, there exists a unique subgame-perfect equilibrium. The value function* $\Pi_t^f(\mathbf{K};\Delta)$ *converges to a limit* $\Pi_t^f(\mathbf{K})$ *that solves the differential equation*

$$\dot{\Pi}_t^f(\mathbf{K}) = -\lambda_t \quad \Bigg(\sum_{j\in\mathcal{J}_f} s_j(\mathbf{p}^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t))\big(p_j^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}_t) - \underbrace{(\Pi_t^f(\mathbf{K}) - \Pi_t^f(\mathbf{K}-\mathbf{e}_j))}_{=\omega_{j,t}^f(\mathbf{K})}\big) \\ - \sum_{j'\neq\mathcal{J}_f} s_{j'}(\mathbf{p}^*(\Omega_t(\mathbf{K});\boldsymbol{\theta}))\big(\underbrace{\Pi_t^f(\mathbf{K}) - \Pi_t^f(\mathbf{K}-\mathbf{e}_{j'})}_{=\omega_{j',t}^f(\mathbf{K})}\big)\Bigg)$$

with boundary conditions $\Pi_t^f(\mathbf{K}) = \mathbf{0}$ *if* $K_j = 0$ *for all* $j \in \mathcal{J}_f$ *and* $\Pi_T^f(\mathbf{K}) = 0$*.*

In order to see that Assumptions 1-4 can be satisfied for all stage games, consider a logit demand specification

$$s_j(\mathbf{p};\boldsymbol{\theta}_t) = \frac{e^{\frac{\delta_j - \alpha_t p_j}{\sigma}}}{1 + \sum_{j'} e^{\frac{\delta_j - \alpha p_j}{\sigma}}} \tag{12}$$

where $\sigma > 0$ is the scaling factor. Then, for $\sigma \to 0$, we are in the Bertrand competition case and as $\sigma \to \infty$, we have perfectly differentiated products.

Lemma 10. *For a logit demand specification* (12)*, holding everything else fixed:*

- *there exists a* $\bar{\sigma}$ *and a* $\bar{\Delta} > 0$ *so that for all* $\sigma > \bar{\sigma}$ *and* $\Delta < \bar{\Delta}$*, the cost matrix* $\Omega_t(\mathbf{K})$ *satisfies Assumption 4 for all* $t \in [0, T]$ *and* $\mathbf{K} \leq \mathbf{K}_0$.
- *there exists a* $\bar{t} < T$ *and* $\bar{\Delta} > 0$ *so that the opportunity cost matrix* $\Omega_t(\mathbf{K})$ *satisfies Assumption 4 for all* $\Delta < \bar{\Delta}$ *and* $t > \bar{t}$,

A.2.3 Price Dynamics

Assume that λ_t and $s_{f,t}$ is independent of time, i.e., $\lambda_t = \lambda$, $\boldsymbol{\theta}_t = \boldsymbol{\theta}$.

Proposition 4. *For* $\mathbf{K}$ *with* $\underline{K} := \min_j K_j$*, the following holds:*

$$p_{j,t}(\mathbf{K}) = \mathcal{O}(|T-t|^{\underline{K}}),\ t \to T.$$

If $(\Pi_t^f)^{(\underline{K})}(\mathbf{K} - \mathbf{e}_{j'}) \neq 0$ *for all* f *and* j' *with* $K_{j'} = \underline{K}$*, then*

$$p_{j,t}(\mathbf{K}) = \Theta(|T-t|^{\underline{K}}),\ t \to T.$$

The proposition shows that prices are more different from 0 close to the deadline the smaller the minimum inventory of products $K_j = \underline{K}$ is. If firms have the same capacity, then any sale leads to price jump. This leads to strong incentives to get out of this state by offering low prices — possibly even prices smaller than the competitive price.

B Proofs

[Appendix B is work in progress]

B.1 Technical results

B.1.1 Continuous time limit

We show convergence of various games in this paper using the following Lemma.

Lemma 11. *Consider a discrete time game where the payoffs of the players $f \in \mathscr{F}$ is described by difference equations of the form*

$$\left(\frac{\Pi^f_{t+\Delta}(\mathbf{K};\Delta)-\Pi^f_t(\mathbf{K};\Delta)}{\Delta}\right)_f = -\lambda_t \mathbf{A}\left(\left(p^*_f\big((\omega^f_{j,t}(\mathbf{K};\Delta))_j,\alpha_t\big)\right)_f, \left(\omega^f_{j,t}(\mathbf{K};\Delta)\right)_{f,j}\right)$$

where $\omega^f_{j,t}(\mathbf{K};\Delta) := \mathbf{\Pi}^f_{t+\Delta}(\mathbf{K};\Delta) - \mathbf{\Pi}^f_{t+\Delta}(\mathbf{K}-\mathbf{e}_j;\Delta)$*,* p^*_f *is continuous in both variables, and* $\mathbf{A}$ *is bounded and continuous in both variables. Then,* $(\Pi^f_t(\mathbf{K};\Delta))_f$ *converges to a limit* $(\Pi^f_t(\mathbf{K}))_f$ *that satisfies*

$$\left(\dot{\Pi}^f_t(\mathbf{K})\right)_f = -\lambda_t \mathbf{A}\left(\left(p^*_f\big((\omega^f_{j,t}(\mathbf{K}))_j\big),\alpha_t\right)_f, \left(\omega^f_{j,t}(\mathbf{K})\right)_{f,j}\right).$$

Proof. Since $\mathbf{A}$ is bounded, the difference equations show that $\mathbf{\Pi}(\Delta) := (\Pi^f(\mathbf{K};\Delta))_{j\in\mathscr{J},\mathbf{K}\le\mathbf{K}_0}$ is equicontinuous and equibounded in t as $\Delta \to 0$. Hence, by the Arzela-Ascoli Theorem, there exist limit points $\mathbf{\Pi}$. We claim that

$$\left(\Pi^f_t(\mathbf{K})\right)_f = \int_t^T \lambda_u \mathbf{A}\left(\left(p^*_f\big((\omega^f_{j,u}(\mathbf{K}))_j,\alpha_u\big)\right)_f, \left(\omega^f_{j,u}(\mathbf{K})\right)_{f,j}\right) du. \tag{13}$$

To this end, we note that

$$\left(\Pi^f_t(\mathbf{K};\Delta)\right)_f = \int_t^T \lambda_{\lceil u\rceil_\Delta} \mathbf{A}\left(\left(p^*_f\big((\omega^f_{j,\lceil u\rceil_\Delta}(\mathbf{K};\Delta))_j,\alpha_{\lceil u\rceil_\Delta}\big)\right)_f, \left(\omega^f_{j,\lceil u\rceil_\Delta}(\mathbf{K};\Delta)\right)_{f,j}\right) du. \tag{14}$$

We take the limit $\Delta \to 0$ on both sides. The left-hand side of (14) converges to the left-hand side of (13). On the right-hand side, $\left(\omega^f_{j,\lceil u\rceil_\Delta}(\mathbf{K};\Delta)\right)_{f,j}$ converges to $\left(\omega^f_{j,u}(\mathbf{K})\right)_{f,j}$. This, and the continuity of $\mathbf{p}^* := (p^*_f)_f$ and $\mathbf{A}$ show that the integrand in (14) converges to the integrand in (13). The dominated convergence theorem finishes the proof. ■

B.1.2 Continuity of stage game prices

Lemma 12. *Let* $g : (\mathbf{q};\boldsymbol{\theta}) \mapsto \mathbf{p}$ *be defined as a function* $\mathbb{R}^J \times \mathbb{R}^I \supset \mathscr{P} \times \Theta \to \mathscr{P}$*, where* g *is continuously differentiable in* $\mathbf{q}$ *and continuous in* $\boldsymbol{\theta}$*,* $\mathscr{P}$ *is compact and convex and* Θ *is path-connected. If* $\det(D_{\mathbf{q}} g(\mathbf{q};\boldsymbol{\theta}) - I) \neq 0$ *for all* $(\mathbf{q};\boldsymbol{\theta}) \in \mathscr{P} \times \Theta$*, then there is a unique* $\mathbf{p}^*(\boldsymbol{\theta})$ *satisfying* $g(\mathbf{p}^*(\boldsymbol{\theta});\boldsymbol{\theta}) = \mathbf{p}^*(\boldsymbol{\theta})$ *and it depends continuously on* $\boldsymbol{\theta}$*.*

Proof. The existence and uniqueness of $\mathbf{p}^*(\boldsymbol{\theta})$ follows directly from Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010). To show continuity, we consider a sequence $(\boldsymbol{\theta}_n)_{n\geq 1}$ converging to some $\boldsymbol{\theta}_\infty$. Thanks to path-connectedness of Θ there exists a continuous path $\mathbf{r} : [0,1] \to \Theta$ and a sequence $a_n \uparrow 1$ such that $\mathbf{r}(a_n) = \boldsymbol{\theta}_n$ and $\mathbf{r}(1) = \boldsymbol{\theta}_\infty$. By Browder's Theorem (Theorem 1.1 in Solan and Solan (2021)), the set $\{(\mathbf{p}^*(\mathbf{r}(a)); a) : a \in [0,1]\} \subset \mathscr{P} \times [0,1]$ is connected. By the main theorem of connectedness, each set $\{(p_j^*(\mathbf{r}(a)); a) : a \in [0,1]\} \subset \mathbb{R} \times [0,1]$ is connected, for all j. By Burgess (1990), the function $a \mapsto p_j^*(\mathbf{r}(a))$ is continuous, so $p_j^*(\boldsymbol{\theta}_n) = p_j^*(\mathbf{r}(a_n)) \to p_j^*(\mathbf{r}(1)) = p_j^*(\boldsymbol{\theta}_\infty)$. ■

B.2 Proofs of the single-firm problem

B.3 Proof of Lemma 1

Note that the profit-maximizing prices of the stage game $\mathbf{p}^M(\boldsymbol{\omega};\boldsymbol{\theta})$ are implicitly given by (4) and

$$g(\mathbf{q};\boldsymbol{\theta}) := \boldsymbol{\omega} - \underbrace{(D_{\mathbf{p}} \mathbf{s}_t(\mathbf{q}))^{-1} \mathbf{s}_t(\mathbf{q})}_{\leq 0}$$

is continuously differentiable in $\mathbf{q}$ and $\boldsymbol{\theta}$ by Assumption 1, and any fixed point must satisfy $\mathbf{q} \geq \boldsymbol{\omega}$ and $\mathbf{q} \leq \boldsymbol{\omega} + \mathbf{1}\bar{\epsilon}$ by Assumption **??** iii). Hence, the convergence to 4 follows by Lemma 11.

B.4 Proof of Proposition 1

Proof. i) To see that Π_t^M is decreasing in t, note that in (4), p_j can always be chosen so that objective function in the maximum is positive. Hence, $\dot{\Pi}_t^M(\mathbf{K}) < 0$.

Next, we show that $\Pi_t^M(\boldsymbol{K}) > \Pi_t^M(\boldsymbol{K} - \boldsymbol{e}_j)$ for all j by induction in $\sum_j K_j$.

Induction start: It is immediate that $\Pi_t^M(\mathbf{e}_j) \geq \Pi_t^M(\mathbf{0}) = 0$ for all j and $t \leq T$.

Induction hypothesis: Assume that $\Pi_t^M(\boldsymbol{K}) > \Pi_t^M(\boldsymbol{K} - \mathbf{e}_j)$ for all $\mathbf{K}$ such that $\sum_j K_j = \bar{K}$.

Induction step: Now, consider a capacity vector $\mathbf{K}$ with $\sum_j K_j = \bar{K} + 1$. By sub-optimality of the prices $\mathbf{p}^M(\boldsymbol{\omega}_t^M(\mathbf{K} - \mathbf{e}_{\bar{j}}))$ given capacity vector $\mathbf{K}$, we have

$$\Pi_t^M\big(\mathbf{K}\big) \geq \int_t^T \lambda_z \Big[\sum_j s_{j,z}\big(\mathbf{p}^M(\boldsymbol{\omega}_z^M(\mathbf{K} - \mathbf{e}_{\bar{j}}))\big)\big(p_{j,z}^M(\boldsymbol{\omega}_z^M(\mathbf{K} - \mathbf{e}_{\bar{j}})) + \Pi_z^M(\mathbf{K} - \mathbf{e}_j)\big) \cdot e^{-\int_t^z \lambda_u \sum_{j''} s_{j'',u}(\mathbf{p}_u^M(\boldsymbol{\omega}_t^M(\mathbf{K})))du} dz > \Pi_t^M\big(\mathbf{K} - \mathbf{e}_{\bar{j}}\big)$$

where the last inequality follows form $\Pi_z^M(\mathbf{K} - \mathbf{e}_j) > \Pi_z^M(\mathbf{K} - \mathbf{e}_j - \mathbf{e}_{\bar{j}})$ by the induction hypothesis.

ii) Next, we show that $\Pi_t^M(\mathbf{K}) - \Pi_t^M(\mathbf{K} - \mathbf{e}_j) \leq \Pi_t^M(\mathbf{K} - \mathbf{e}_j) - \Pi_t^M(\mathbf{K} - 2\mathbf{e}_j)$ for all j. To this end, let

$$H(\mathbf{x}; \boldsymbol{\theta}) = -\max_{\mathbf{p}} \sum_j s_j(\mathbf{p}; \boldsymbol{\theta})(p_j - x_j).$$

Note that H is concave as a minimum of affine functions, strictly increasing in $\mathbf{x}$, and $H(\mathbf{0}; \boldsymbol{\theta}) = 0$ by Assumption **??** iii). Since H is concave, it admits the representation

$$H(\mathbf{x}; \boldsymbol{\theta}) = \inf_{\mathbf{s}} (\mathbf{s} \cdot \mathbf{x} - H^*(\mathbf{s}; \boldsymbol{\theta}))$$

where the concave $H^*(\mathbf{s}; \boldsymbol{\theta}) = \inf_{\mathbf{x}} (\mathbf{x} \cdot \mathbf{s} - H(\mathbf{x}; \boldsymbol{\theta}))$ is the concave conjugate of H, with

$H^{*}(\mathbf{0};\boldsymbol{\theta})=0$. Moreover,

$$\dot{\Pi}_t^M(\mathbf{K})=\lambda_t H(\nabla\Pi_t(\mathbf{K});\boldsymbol{\theta}_t)$$

where $\nabla\Pi_t^M(\mathbf{K})=\left(\Pi_t^M(\mathbf{K})-\Pi_t^M(\mathbf{K}-\mathbf{e}_j)\right)_j$. Thus, $\Pi_t^M(\mathbf{K})$ is the value function for the optimal control problem

$$\Pi_t^M(\mathbf{K})=\sup_{\mathbf{s}\in\mathscr{A}}\mathbb{E}\left[\int_t^T \lambda_u H^{*}(\mathbf{s}_u;\boldsymbol{\theta}_u)du\,\middle|\,\mathbf{X}_t^{\mathbf{s}}=\mathbf{K}\right]=:\sup_{\mathbf{s}} J_t(\mathbf{K},\mathbf{s})$$

where $\mathbf{X}_t^{\mathbf{a}}$ is the process which jumps by $-\mathbf{e}_j$ at rate $\lambda_t s_{j,t}$ and $\mathbf{s}\in\mathscr{A}$ are processes adapted with respect to $\mathbf{X}^{\mathbf{s}}$, with the property $s_{j,t}=0$ if $X_{j,t}^{\mathbf{s}}=0$ (Theorem 8.1 in Fleming and Soner (2006)). Let $\mathbf{s}_{\mathbf{K}}^{*}$ be the optimal control in the previous equation and $\mathbf{s}_{K-2}^{*}$ be the optimal control when $\mathbf{K}$ is replaced by $\mathbf{K}-2\mathbf{e}_j$. Then, note that since $\mathbf{s}_{\mathbf{K}}^{*},\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^{*}\in\mathscr{A}$, $\frac{\mathbf{s}_{\mathbf{K}}^{*}+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^{*}}{2}\in\mathscr{A}$ because the process $\left(\mathbf{X}_s^{\frac{\mathbf{s}_{\mathbf{K}}^{*}+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f}^{*}}{2}}\right)_s$ can be chosen as $\left(\frac{\mathbf{X}_s^{\mathbf{s}_{\mathbf{K}}^{*}}+\mathbf{X}_s^{\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f}^{*}}}{2}\right)_s$ ("coupling argument"). Hence,

$$\begin{aligned}
&\Pi_t^M(\mathbf{K})+\Pi_t^M(\mathbf{K}-2\mathbf{e}_j)-2\Pi_t^M\left(\mathbf{K}-\mathbf{e}_j\right) &&\leq\\
&J_t(\mathbf{K},s_{\mathbf{K}}^{*})+J_t(\mathbf{K}-2\mathbf{e}_j,s_{\mathbf{K}-2\mathbf{e}_j}^{*})-2\,J_t\left(\mathbf{K}-\mathbf{e}_j,\frac{\mathbf{s}_{\mathbf{K}}^{*}+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^{*}}{2}\right) &&\leq\\
&\mathbb{E}\left[\int_t^T\lambda_u\left(H^{*}(\mathbf{s}_{\mathbf{K},u}^{*})+H^{*}(\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j,u}^{*})-2H^{*}\left(\frac{\mathbf{s}_{\mathbf{K},u}^{*}+\mathbf{s}_{\mathbf{K}-2\mathbf{e}_f,u}^{*}}{2}\right)\right)du\,\middle|\,\mathbf{X}_t^{\mathbf{s}_{\mathbf{K}}^{*}}=\mathbf{K},\,\mathbf{X}_t^{\mathbf{s}_{\mathbf{K}-2\mathbf{e}_j}^{*}}=\mathbf{K}-2\mathbf{e}_j,\right] &&\leq 0.
\end{aligned}$$

iii) To show that $\omega_{j,t}^M(\mathbf{K}_t)$ is a submartingale, we show that for any capacity vector $\mathbf{K}$,

$$\lim_{\Delta\to 0}\frac{\mathbb{E}_0\left[\omega_{j,t+\Delta}^M(\mathbf{K}_{t+\Delta})-\omega_{j,t}^M(\mathbf{K}_t)\middle|\mathbf{K}_t=\bar{\mathbf{K}}\right]}{\Delta}\geq 0.$$

To this end, first, note that $\mathbf{K}_t$ is right-continuous in t. Further, for $\mathbf{K}$ with $K_j=0$, we set $\omega_{j,t}^M(\mathbf{K})=\infty$ for all t. Thus, we are setting the opportunity cost of selling a unit if no capacity is left to infinity, which is equivalent to the constraint of not being able to sell units that are not available.

Then, we have for $\bar{\mathbf{K}}$ with $\bar{K}_j = 1$ that

$$\lim_{\Delta\to 0} \frac{\mathbb{E}_0\big[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t = \bar{\mathbf{K}}]}{\Delta} > 0.$$

Next consider $\bar{\mathbf{K}}$ with $\bar{K}_j \geq 0$. Then, we have that

$$\begin{aligned}
&\lim_{\Delta\to 0} \frac{\mathbb{E}_0\big[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t = \bar{\mathbf{K}}]}{\Delta} = \\
&\lim_{\Delta\to 0} \frac{\mathbb{E}_0\big[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_{t+\Delta})|\mathbf{K}_t = \bar{\mathbf{K}}]}{\Delta} + \lim_{\Delta\to 0} \frac{\mathbb{E}_0\big[\omega^M_{j,t}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_t)|\mathbf{K}_t = \bar{\mathbf{K}}]}{\Delta} = \\
&\dot{\omega}^M_{j,t}(\bar{\mathbf{K}}) + \lambda_t \sum_{j'} s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\big(\omega^M_{j,t}(\bar{\mathbf{K}} - \mathbf{e}_{j'}) - \omega^M_{j,t}(\bar{\mathbf{K}})\big)
\end{aligned}$$

by right-continuity of the process K_t. By (4), we can write

$$\dot{\omega}^M_{j,t}(\bar{\mathbf{K}}) \;=\; -\lambda_t\Big[\sum_{j'} s_{j',t}(\mathbf{p}^M_t(\bar{\mathbf{K}}))\big(p^M_{j',t}(\bar{\mathbf{K}}) - \omega^M_{j',t}(\bar{\mathbf{K}})\big) - s_{j',t}(p^M_t(\bar{\mathbf{K}} - \mathbf{e}_j))\big(p^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j) - \omega^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j)\big)\Big].$$

and we know that

$$\begin{aligned}
-\omega^M_{j',t}(\bar{\mathbf{K}}) + \omega^M_{j,t}(\bar{\mathbf{K}}) - \omega^M_{j,t}(\bar{\mathbf{K}} - \mathbf{e}_{j'}) \;&=\; \Pi^M(\bar{\mathbf{K}} - \mathbf{e}_{j'}) - \Pi^M(\bar{\mathbf{K}} - \mathbf{e}_j) - \Pi^M(\bar{\mathbf{K}} - \mathbf{e}_{j'}) + \Pi^M(\bar{\mathbf{K}} - \mathbf{e}_{j'} - \mathbf{e}_j) \\
&=\; \omega^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j)
\end{aligned}$$

Hence, $\lim_{\Delta\to 0} \frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta}) - \omega^M_{j,t}(\mathbf{K}_t)|_t = \bar{\mathbf{K}}\right]}{\Delta}$ is equal to

$$-\lambda_t\Big[\sum_{j'} s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}})\big)\big(p^M_{j',t}(\bar{\mathbf{K}}) - \omega^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j)\big) - s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}} - \mathbf{e}_j)\big)\big(p^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j) - \omega^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j)\big)\Big]$$

Then, note that by definition of $\mathbf{p}^M_t(\bar{\mathbf{K}} - \mathbf{e}_j)$,

$$\sum_{j'} s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}})\big)\big(p^M_{j',t}(\bar{\mathbf{K}}) - \omega^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j)\big) \leq \sum_{j'} s_{j',t}\big(\mathbf{p}^M_t(\bar{\mathbf{K}} - \mathbf{e}_j)\big)\big(p^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_j)) - \omega^M_{j',t}(\bar{\mathbf{K}} - \mathbf{e}_{j'})\big).$$

Hence, $\lim_{\Delta\to 0} \frac{\mathbb{E}_0\left[\omega^M_{j,t+\Delta}(\mathbf{K}_{t+\Delta})-\omega^M_{j,t}(\mathbf{K}_t)|_t=\bar{\mathbf{K}}\right]}{\Delta} \geq 0$. ■

B.5 Proof of Lemma 2

First, we show that there exists a $\bar{p} < \infty$ so that for any any vector of prices $\mathbf{q}$, the best response price p_j for any product j is bounded by $\bar{p}$. We proceed with a proof by contradiction.

Assume that there is an increasing sequence of $\bar{b}^n \to_{n\to\infty}$ such that there is a vector of prices $\mathbf{q}^n$ such that there is a best response price $p^n_j > \bar{b}^n$.

$$
\begin{aligned}
0 \leq \quad & \underbrace{\frac{\partial s_j}{\partial p_j}}_{<0}(p_j-\omega^f_j)+\sum_{k\in\mathscr{J}_f\setminus\{j\}}\underbrace{\frac{\partial s_k}{\partial p_j}}_{>0}(p_k-\omega^f_k) \\
& -\sum_{k\notin\mathscr{J}_f}\frac{\partial s_k}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\omega^f_k+s_j(\mathbf{q}^{-f},\mathbf{p}^f) \\
\leq \quad & \left|\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\right|\left[\frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega^f_k)_+-\left(2(p_j-\omega^f_j)+\underbrace{\frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)}}_{\geq -\bar{\epsilon}\text{ by Assumption ??-??}}\right)\right. \\
& \left.+\left(\sum_{k\notin\mathscr{J}_f}\frac{|\omega^f_k|}{J-1}\right)\right].
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
& \forall j\in\mathscr{J}_f: \quad 2(p_j-\omega^f_j)+\frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)} \leq \frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega^f_k)_+ +\sum_{k\notin\mathscr{J}_f}\frac{|\omega^f_k|}{J-1} \\
\Rightarrow \quad & \forall j\in\mathscr{J}_f: \quad 2(p_j-\omega^f_j)-\bar{\epsilon} \leq \frac{1}{J-1}\sum_{k\in\mathscr{J}_f\setminus\{j\}}(p_k-\omega^f_k)_+ +\sum_{k\notin\mathscr{J}_f}\frac{|\omega^f_k|}{J-1}.
\end{aligned}
$$

$$
\begin{aligned}
0 \leq \quad & \underbrace{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^{f,*}_{-f},p_j)(p_j-\omega^f_j)}_{<0} + \sum_{k\in\mathscr{J}_f\setminus\{j\}} \underbrace{\frac{\partial s_k}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)(p_k-\omega^f_k)}_{>0} \\
& - \sum_{k\notin\mathscr{J}_f} \frac{\partial s_k}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\omega^f_k + s_j(\mathbf{q}^{-f},\mathbf{p}^f) \\
\leq \quad & \left|\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)\right| \Bigg[\frac{1}{J-1} \sum_{k\in\mathscr{J}_f\setminus\{j\}} (p_k-\omega^f_k)_+ - \Bigg(2(p_j-\omega^f_j) + \underbrace{\frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)}}_{\geq -\bar{\epsilon} \text{ by Assumption ??-??}} \Bigg) \\
& + \left(\sum_{k\notin\mathscr{J}_f} \frac{|\omega^f_k|}{J-1} \right) \Bigg].
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
& \forall\, j \in \mathscr{J}_f: \quad 2(p_j-\omega^f_j) + \frac{s_j(\mathbf{q}^{-f},\mathbf{p}^f)}{\frac{\partial s_j}{\partial p_j}(\mathbf{q}^{-f},\mathbf{p}^f)} \leq \frac{1}{J-1} \sum_{k\in\mathscr{J}_f\setminus\{j\}} (p_k-\omega^f_k)_+ + \sum_{k\notin\mathscr{J}_f} \frac{|\omega^f_k|}{J-1} \\
\Rightarrow \quad & \forall\, j \in \mathscr{J}_f: \quad 2(p_j-\omega^f_j) - \bar{\epsilon} \leq \frac{1}{J-1} \sum_{k\in\mathscr{J}_f\setminus\{j\}} (p_k-\omega^f_k)_+ + \sum_{k\notin\mathscr{J}_f} \frac{|\omega^f_k|}{J-1}.
\end{aligned}
$$

Using this for $j = l$ maximizing the left-hand side we obtain a contradiction once we choose $\bar{p}$ sufficiently large. Thus, the best response of each firm for each product is strictly smaller than a constant $\bar{p}$.

Hence, by Assumption **??**, there is a unique fixed point of $g(\mathbf{p})$ by Lemma 2 (Kellogg (1976)) in Konovalov and Sándor (2010).

B.6 Proof of Lemma 3

Note that the profit-maximizing prices of the stage game $\mathbf{p}^M(\boldsymbol{\omega};\boldsymbol{\theta})$ are implicitly given by (4) and

$$
g(\mathbf{q};\boldsymbol{\theta}) := \boldsymbol{\omega} - \underbrace{(D_{\mathbf{p}}\mathbf{s}_t(\mathbf{q}))^{-1}\mathbf{s}_t(\mathbf{q})}_{\leq 0}
$$

is continuously differentiable in $\mathbf{q}$ and $\boldsymbol{\theta}$ by Assumption 1 i), and any fixed point must satisfy $\mathbf{q} \geq \boldsymbol{\omega}$ and $\mathbf{q} \leq \boldsymbol{\omega} + \mathbf{1}\bar{\epsilon}$ by Assumption **??** iii). Hence, the convergence to 4 follows by Lemma 11.

Recall that the first-order conditions of firm f's payoff with respect to product $j \in \mathscr{J}_f$ are given by

$$p_j - \left(\omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} (p_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} \omega_{j'}^f \right) = -\frac{s_j(\mathbf{p})}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}}.$$

Further, the observation that $\frac{\partial s_j(\mathbf{p})}{\partial p_j} = -\sum_{j'' \in \mathscr{J}^0 \setminus \{j\}} \frac{\partial s_{j''}(\mathbf{p})}{\partial p_j}$, and by Assumption **??** (Independence of Irrelevant alternatives) it follows that

$$\begin{aligned}
c(\mathbf{p}_{-j}; \boldsymbol{\omega}) :=& \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} (p_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\frac{\partial s_j(\mathbf{p})}{\partial p_j}} \omega_{j'}^f \\
=& \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \frac{\frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}}{\sum_{j' \in \mathscr{J}^0 \setminus \{j\}} \frac{\partial s_{j'}(\mathbf{p})}{\partial p_j}} (p_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \frac{\frac{s_{j'}(\mathbf{p})}{\partial p_j}}{\sum_{j'' \in \mathscr{J}^0 \setminus \{j\}} \frac{\partial s_{j''}(\mathbf{p})}{\partial p_j}} \omega_{j'}^f \\
=& \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \tilde{s}_{j,j'}(\mathbf{p}_{-j})(p_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \tilde{s}_{j,j'}(\mathbf{p}_{-j}) \omega_{j'}^f.
\end{aligned}$$

Thus, the first-order conditions of the stage game are equivalent to the first order conditions of a game with $\mathscr{J}$ players where each player j's payoff is given by

$$s_j(\mathbf{p})(p_j - c(\mathbf{p}_{-j}; \boldsymbol{\omega})) + d(\mathbf{p}_{-j}; \boldsymbol{\omega}).$$

B.7 Proof of Lemma 7

Assume $s_j(\mathbf{p}; \boldsymbol{\theta}) > 0$ satisfies Assumptions **??** and **??**. Then, we define the best-response function of "player" j in the game defined in Lemma 3 by

$$\mathscr{R} : \quad \mathbf{q} \mapsto \left(\arg\max_{p_j} s_j(\mathbf{q})(p_j - c_j(\mathbf{q}_{-j}; \boldsymbol{\omega}, \boldsymbol{\theta})) + d_j(\mathbf{q}_{-j}; \boldsymbol{\omega}, \boldsymbol{\theta}) \right)_{j \in \mathscr{J}}.$$

where for $\tilde{s}_{j,j'}(\mathbf{q}_{-j}) := \frac{\frac{\partial s_{j'}}{\partial p_j}(\mathbf{q})}{\sum\limits_{j'' \in \mathscr{J}^0 \setminus \{j\}} \frac{\partial s_{j''}}{\partial p_j}(\mathbf{q})}$ and $j \in \mathscr{J}_f$

$$c_j(\mathbf{q}_{-j};\boldsymbol{\omega},\boldsymbol{\theta}) := \omega_j^f - \sum_{j' \in \mathscr{J}_f \setminus \{j\}} \tilde{s}_{j,j'}(\mathbf{q}_{-j})(q_{j'} - \omega_j^f) + \sum_{j' \notin \mathscr{J}_f} \tilde{s}_{j,j'}(\mathbf{q}_{-j})\omega_{j'}^f. \tag{15}$$

First, we show that $\mathscr{R}$ is well-defined as a function $\mathbb{R}^{\mathscr{J}} \mapsto [-\infty,\infty]^{\mathscr{J}}$ (rather than a correspondence). To this end, note that player j's profit is increasing in p_j if and only if

$$p_j - c_j(\mathbf{q}_{-j};\boldsymbol{\omega},\boldsymbol{\theta}) + \underbrace{\frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}}_{<0 \text{ by Assumption } \textbf{??}} \leq 0 \tag{16}$$

and the left-hand side is increasing in p_j by Assumption **??**.

Then, note that the best-response function $\mathscr{R}$ takes values in $[\underline{p},\overline{p}]^{\mathscr{J}}$, with $\underline{p} > -\infty$ and $\overline{p} < \infty$, for all $\mathbf{q}$ by the same argument as in the proof of Lemma 2.

Now, consider $\mathscr{R} : [\underline{p},\overline{p}]^{\mathscr{J}} \to [\underline{p},\overline{p}]^{\mathscr{J}}$. In order to show continuitiy of $\mathscr{R}$, we use the implicit function theorem in the form of Theorem 1.A.4 in Dontchev and Rockafellar (2009). To this end, for $\epsilon > 0$, consider the mapping

$$\Phi : (\mathbf{p},\mathbf{q}) \mapsto \left(p_j - \epsilon\big(p_j - c_j(\mathbf{q}_{-j};\Omega,\boldsymbol{\theta}) + \frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}\big)\right)_{j \in \mathscr{J}}$$

Then $D_{\mathbf{p}}\Phi$ is a diagonal matrix with diagonal entries

$$\phi_j := 1 - \epsilon\left(1 + \underbrace{\frac{\partial}{\partial p_j} \frac{s_j(\mathbf{q}_{-j},p_j)}{\frac{\partial s_j(\mathbf{q}_{-j},p_j)}{\partial p_j}}}_{\geq 0 \text{ by Assumption } \textbf{??}}\right)$$

Let $\epsilon > 0$ be so that $\phi_j > 0$ for all j. Then all diagonal entries are in $(0,1-\epsilon)$ and Φ is Lipschitz continuous with Lipschitz constant $\max_j \phi_j$. Further $D_{\mathbf{q}}\Phi$ is bounded because it is continuous and the function is defined on a compact set $[\underline{p},\overline{p}]^{\mathscr{J}}$. Thus, $\mathscr{R}$ is continuous. Hence, by Brouwer's fixed-point theorem $\mathscr{R} : [\underline{p},\overline{p}]^{\mathscr{J}} \to [\underline{p},\overline{p}]^{\mathscr{J}}$ has a fixed point.

B.8 Dynamics

xxx to be completed xxx

Assume that λ_t and $s_{f,t}$ is independent of time, i.e., $\lambda_t = \lambda$, $\alpha_t = \alpha$. For t close to T, we know from Lemma 2 that the equilibirum of the stage game is unique and the price vectors $p_t^*(\mathbf{K})$ are implicitly defined by a system of equations given by

$$D_{\mathbf{p}^f} \mathbf{s}^f(\mathbf{p}_t^*(\mathbf{K}))\mathbf{p}_t^{f,*}(\mathbf{K}) - \mathbf{s}(\mathbf{p}_t^*(\mathbf{K}))\boldsymbol{\omega}_t^f(\mathbf{K}) = 0$$

for all f. The only time-dependent variables are then $\Omega_t(\mathbf{K}) = (\boldsymbol{\omega}_t^f)_{f\in\mathscr{F}}$. The n-th time derivative $(p_t^*)^{(n)}(\mathbf{K})$ depends on the time derivatives $\Omega_t(\mathbf{K}), \ldots, \Omega_t^{(n)}(\mathbf{K})$.

We are interested in the limit as $t \to T$. First, $\lim_{t\to T} \Omega_t = 0$. Furthermore, we can write

$$\begin{aligned}
\dot{\omega}_{j,t}^f(\mathbf{K}) =& \dot{\Pi}_t^f(\mathbf{K}) - \dot{\Pi}_t^f(\mathbf{K} - \mathbf{e}_j) \\
=& -\lambda\big[\mathbf{s}^f(\mathbf{p}_t^*(\mathbf{K}))\mathbf{p}_t^{f,*}(\mathbf{K}) - \mathbf{s}(\mathbf{p}_t^*(\mathbf{K}))\boldsymbol{\omega}_t^f(\mathbf{K}) - (\mathbf{s}^f(\mathbf{p}_t^*(\mathbf{K}-\mathbf{e}_j))\mathbf{p}_t^{f,*}(\mathbf{K}-\mathbf{e}_j) - \mathbf{s}(\mathbf{p}_t^*(\mathbf{K}-\mathbf{e}_j))\boldsymbol{\omega}_t^f(\mathbf{K}-\mathbf{e}_j))\big]
\end{aligned}$$

Thus, as $t \to T$, $\dot{\omega}_{j,t}^f(\mathbf{K}) = 0$ if $K_j > 1$. If $j \in \mathscr{J}_f$ and $K_j = 1$, then $\dot{\omega}_{j,t}^f(\mathbf{K}) < 0$. If $j \notin \mathscr{J}^f$ and $K_j = 1$, then by the competition effect $\dot{\omega}_{j,t}^f(\mathbf{K}) > 0$.

This implies that $\dot{p}_{j,T}^*(\mathbf{K}) < 0$ if $K_j = 1$ and $\dot{p}_{j,T}^*(\mathbf{K}) = 0$ otherwise.

Induction assumption: If $K_j > n-1$ for all j, then as $t \to T$, $(\omega_{j,t}^f)^{(n-1)}(\mathbf{K}) = 0$ for all f, j.

We can also calculate all other time derivatives recursively

$$(\omega_{j,t}^f)^{(n)}(\mathbf{K}) = -\lambda\big[G^n((\Omega_t^{(m)}(\mathbf{K}))_{m=0}^{n-1}) - G^n(\Omega_t^{(m)}(\mathbf{K}-\mathbf{e}_j))_{m=0}^{n-1}))\big].$$

Then, note if $\min_i K_i > n$, then $(\omega_{j,t}^f)^{(n)}(\mathbf{K}) = 0$. If $\min_i K_i = n$, then

$$(\omega_{j,t}^f)^{(n)}(\mathbf{K}) = -\lambda\big[-G^{(n)}(\Omega_t^{(m)}(\mathbf{K}-\mathbf{e}_j))_{m=0}^{n-1}))\big] = -(\Pi_t^f)^{(n)}(\mathbf{K}-\mathbf{e}_j).$$

C Classic logit and nested logit calculations

C.1 Classic logit demand

In this section, we consider a logit demand model parametrized by $\boldsymbol{\theta} = ((\delta_j)_{j\in\mathscr{J}}, \alpha)$, given by

$$s_j(\mathbf{p};\boldsymbol{\theta}) = \frac{e^{\frac{\delta_j - \alpha p_j}{\sigma}}}{1 + \sum_{j\in\mathscr{J}} e^{\frac{\delta_j - \alpha p_j}{\sigma}}}.$$

All simulations in this paper and the empirical specification assumes this demand structure. Then,

$$\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{\sigma} s_j(1-s_j) \qquad \frac{\partial s_j}{\partial p_{j'}} = \frac{\alpha}{\sigma} s_j s_{j'}.$$

Then, we have that $\hat{\boldsymbol{\epsilon}} = \frac{1}{\alpha s_0(\mathbf{p};\boldsymbol{\theta})}\mathbf{1}$ and therefore, $\det\left(D_p \hat{\boldsymbol{\epsilon}} - I\right) = (-1)^J \det\left(\frac{1}{s_0(\mathbf{p};\boldsymbol{\theta})}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})\ldots\mathbf{s}(\mathbf{p};\boldsymbol{\theta})) + I\right) = (-1)^J \frac{1}{s_0(\mathbf{p};\boldsymbol{\theta})}$.

First, we show that Assumption **??** is satisfied. To this end, note that

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = \frac{\sigma}{\alpha}\begin{pmatrix} -s_1(1-s_1) & s_1 s_2 & \ldots & s_1 s_J \\ s_2 s_1 & \ddots & & s_2 s_J \\ \vdots & & \ddots & \vdots \\ s_J s_1 & \ldots & s_J s_{J-1} & -s_J(1-s_J) \end{pmatrix}^{-1} = -\frac{\sigma}{\alpha s_0}\begin{pmatrix} 1+\frac{s_0}{s_1} & 1 & \ldots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \ldots & 1 & 1+\frac{s_0}{s_1} \end{pmatrix}.$$

Hence,

$$\hat{\boldsymbol{\epsilon}} = (D_p\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}) = -\frac{\sigma}{\alpha s_0}\begin{pmatrix} 1+\frac{s_0}{s_1} & 1 & \ldots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \ldots & 1 & 1+\frac{s_0}{s_1} \end{pmatrix}\begin{pmatrix} s_1 \\ \vdots \\ s_J \end{pmatrix} = \frac{\sigma}{\alpha s_0}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and since $\frac{\partial}{\partial p_j}\frac{1}{s_0} = -\frac{\alpha}{\sigma}\frac{s_j}{s_0}$,

$$\det\left(D_{\mathbf{p}}\hat{\epsilon} - I\right) = \det\begin{pmatrix} -\frac{s_1}{s_0} - 1 & \dots & -\frac{s_J}{s_0} \\ \vdots & \ddots & \vdots \\ -\frac{s_1}{s_0} & \dots & -\frac{s_J}{s_0} - 1 \end{pmatrix} = (-1)^J \frac{1}{s_0}$$

Next, note that for a for a any $f \in \mathscr{F}$, if we define $s_0^f(\mathbf{p}) = 1 - \sum_{j \in \mathscr{J}_f} s_j(\mathbf{p})$, then

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} = \frac{\sigma}{\alpha}\begin{pmatrix} -s_1(1-s_1) & s_1 s_2 & \dots & s_1 s_{J_f} \\ s_2 s_1 & \ddots & & s_2 s_{J_f} \\ \vdots & & \ddots & \vdots \\ s_{J_f} s_1 & \dots & s_{J_f} s_{J_f - 1} & -s_{J_f}(1 - s_{J_f}) \end{pmatrix}^{-1} = -\frac{\sigma}{\alpha s_0^f}\begin{pmatrix} 1 + \frac{s_0^f}{s_1} & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1 + \frac{s_0^f}{s_1} \end{pmatrix}..$$

Further, $\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega} = \sum_j s_j \omega_j^f$ and

$$D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}) = \frac{\alpha}{\sigma}\begin{pmatrix} s_1\left(\sum_{j \in \mathscr{J}} s_j \omega_j^f - \omega_1^f\right) \\ \vdots \\ s_{J_f}\left(\sum_{j \in \mathscr{J}} s_j \omega_j^f - \omega_{J_f}^f\right) \end{pmatrix}$$

Hence,

$$\begin{aligned} (D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal \boldsymbol{\omega}) &= -\frac{1}{s_0^f}\begin{pmatrix} \left(\sum_{j \in \mathscr{J}_f} s_j + s_0^f\right) \sum_{j \in \mathscr{J}} s_j \omega_j^f - \sum_{j \in \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_1^f \\ \vdots \\ \left(\sum_{j \in \mathscr{J}_f} s_j + s_0^f\right) \sum_{j \in \mathscr{J}} s_j \omega_j^f - \sum_{j \in \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_{J_f}^f \end{pmatrix} \\ &= -\frac{1}{s_0^f}\begin{pmatrix} \sum_{j \in \mathscr{J} \setminus \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_1^f \\ \vdots \\ \sum_{j \in \mathscr{J} \setminus \mathscr{J}_f} s_j \omega_j^f - s_0^f \omega_{J_f}^f \end{pmatrix} \end{aligned}$$

Further, since

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}) == \frac{\sigma}{\alpha s_0^f}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

Finally,

$$g^f(\mathbf{p}) = -\begin{pmatrix}\frac{\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j}{1-\sum_{j\in\mathscr{J}_f} s_j}\omega_j^f - \omega_1^f + \frac{1}{1-\sum_{j\in\mathscr{J}_f} s_j}\frac{\sigma}{\alpha}\\ \vdots\\ \frac{\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j}{1-\sum_{j\in\mathscr{J}_f} s_j}\omega_j^f - \omega_{J_f}^f + \frac{1}{1-\sum_{j\in\mathscr{J}_f} s_j}\frac{\sigma}{\alpha}\end{pmatrix}$$

and

$$D_{\mathbf{p}}g^f(\mathbf{p}) = \left(\begin{pmatrix}\frac{s_k}{1-\sum_{j\in\mathscr{J}_f} s_j}\\ \vdots\\ \frac{s_k}{1-\sum_{j\in\mathscr{J}_f} s_j}\end{pmatrix}_{k\in\mathscr{J}_f}, \begin{pmatrix}\frac{\alpha}{\sigma}\frac{-(1-\sum_{j\in\mathscr{J}_f} s_j)(1-\sum_{j\notin\mathscr{J}_f} s_j)+(\sum_{j\in\mathscr{J}_f} s_j)(\sum_{j\notin\mathscr{J}_f} s_j)}{(1-\sum_{j\in\mathscr{J}_f} s_j)^2}\omega_j^f + s_k\frac{\sum_{j\in\mathscr{J}_f} s_j}{(1-\sum_{j\in\mathscr{J}_f} s_j)^2}\\ \vdots\\ \frac{\alpha}{\sigma}\frac{-(1-\sum_{j\in\mathscr{J}_f} s_j)(1-\sum_{j\notin\mathscr{J}_f} s_j)+(\sum_{j\in\mathscr{J}_f} s_j)(\sum_{j\notin\mathscr{J}_f} s_j)}{(1-\sum_{j\in\mathscr{J}_f} s_j)^2}\omega_j^f + s_k\frac{\sum_{j\in\mathscr{J}_f} s_j}{(1-\sum_{j\in\mathscr{J}_f} s_j)^2}\end{pmatrix}_{k\notin\mathscr{J}_f}\right)$$

For large σ the term in front of ω_j^f vanishes relative to the probability.

C.2 Nested logit demand

In this section, we consider a nested logit demand model given by

$$s_j(\mathbf{p}) = \underbrace{\frac{e^{\frac{\delta_j-\alpha p_j}{1-\sigma}}}{\sum_{j\in\mathscr{J}} e^{\frac{\delta_j-\alpha p_j}{1-\sigma}}}}_{=:s_{j|\mathscr{J}}(\mathbf{p})}\frac{\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}}{1+\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}} \qquad s_0(\mathbf{p}) = \frac{1}{1+\left(\sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}\right)^{1-\sigma}}$$

To simplify notation, let $D_J := \sum_{i\in\mathscr{J}} e^{\frac{\delta_i-\alpha p_i}{1-\sigma}}$. Then,

$$\frac{\partial s_j}{\partial p_j} = -\frac{\alpha}{1-\sigma} s_j \left(1-\left(\sigma s_{j|\mathscr{J}} + (1-\sigma)s_j\right)\right) \qquad \frac{\partial s_j}{\partial p_{j'}} = \frac{\alpha}{1-\sigma} s_{j'} \left(\sigma s_{j|\mathscr{J}} + (1-\sigma)s_j\right).$$

$$(D_{\mathbf{p}}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1} = -\frac{1}{\alpha s_0}\begin{pmatrix} \frac{\sigma}{D_J^{1-\sigma}}+1+(1-\sigma)\frac{s_0}{s_1} & \frac{\sigma}{D_J^{1-\sigma}}+1 & \dots & \frac{\sigma}{D_J^{1-\sigma}}+1 \\ \frac{\sigma}{D_J^{1-\sigma}}+1 & \ddots & & \frac{\sigma}{D_J^{1-\sigma}}+1 \\ \vdots & & \ddots & \vdots \\ \frac{\sigma}{D_J^{1-\sigma}}+1 & \dots & \frac{\sigma}{D_J^{1-\sigma}}+1 & \frac{\sigma}{D_J^{1-\sigma}}+1+(1-\sigma)\frac{s_0}{s_j} \end{pmatrix}$$

Hence,

$$\hat{\boldsymbol{\epsilon}} = (D_p\mathbf{s}(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}(\mathbf{p};\boldsymbol{\theta}) = -\left(\frac{\sigma}{\alpha s_0}+\frac{1-\sigma}{\alpha s_0}\right)\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix} = -\frac{1}{\alpha s_0}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$$

and therefore

$$D_p\hat{\boldsymbol{\epsilon}} - I = -\begin{pmatrix} \frac{1}{D_J}\frac{s_1}{s_0}-1 & \dots & \frac{1}{D_J}\frac{s_J}{s_0} \\ & \ddots & \\ \frac{1}{D_J}\frac{s_1}{s_0} & \dots & \frac{1}{D_J}\frac{s_J}{s_0}-1 \end{pmatrix}.$$

Thus, $\det\left(D_p\hat{\boldsymbol{\epsilon}}-I\right) = (-1)^J\left(\frac{1}{D_J s_0} - \frac{1-D_J}{D_J}\right)$.

Next, note that for a for a any $f\in\mathscr{F}$, if we define $s_0^f(\mathbf{p}) = 1-\sum_{j\in\mathscr{J}_f} s_j(\mathbf{p})$, then

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1} = \frac{\sigma}{\alpha}\begin{pmatrix} -s_1(1-s_1) & s_1 s_2 & \dots & s_1 s_{J_f} \\ s_2 s_1 & \ddots & & s_2 s_{J_f} \\ \vdots & & \ddots & \vdots \\ s_{J_f}s_1 & \dots & s_{J_f}s_{J_f-1} & -s_{J_f}(1-s_{J_f}) \end{pmatrix}^{-1} = -\frac{\sigma}{\alpha s_0^f}\begin{pmatrix} 1+\frac{s_0^f}{s_1} & 1 & \dots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \dots & 1 & 1+\frac{s_0^f}{s_1} \end{pmatrix}..$$

Further, $\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal\boldsymbol{\omega} = \sum_j s_j \omega_j^f$ and

$$D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal\boldsymbol{\omega}) = \frac{\alpha}{\sigma}\begin{pmatrix} s_1\big(\sum_{j\in\mathscr{J}} s_j\omega_j^f - \omega_1^f\big) \\ \vdots \\ s_{J_f}\big(\sum_{j\in\mathscr{J}} s_j\omega_j^f - \omega_{J_f}^f\big)\end{pmatrix}$$

Hence,

$$\begin{aligned}(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1}D_{\mathbf{p}^f}(\mathbf{s}(\mathbf{p};\boldsymbol{\theta})^\intercal\boldsymbol{\omega}) &= -\frac{1}{s_0^f}\begin{pmatrix}\big(\sum_{j\in\mathscr{J}_f} s_j + s_0^f\big)\sum_{j\in\mathscr{J}} s_j\omega_j^f - \sum_{j\in\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_1^f \\ \vdots \\ \big(\sum_{j\in\mathscr{J}_f} s_j + s_0^f\big)\sum_{j\in\mathscr{J}} s_j\omega_j^f - \sum_{j\in\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_{J_f}^f\end{pmatrix} \\ &= -\frac{1}{s_0^f}\begin{pmatrix}\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_1^f \\ \vdots \\ \sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j\omega_j^f - s_0^f\,\omega_{J_f}^f\end{pmatrix}\end{aligned}$$

Further, since

$$(D_{\mathbf{p}^f}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}))^{-1}\mathbf{s}^f(\mathbf{p};\boldsymbol{\theta}) == \frac{\sigma}{\alpha s_0^f}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$$

Finally,

$$g^f(\mathbf{p}) = -\begin{pmatrix}\frac{\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j}{1-\sum_{j\in\mathscr{J}_f} s_j}\omega_j^f - \omega_1^f + \frac{1}{1-\sum_{j\in\mathscr{J}_f} s_j}\frac{\sigma}{\alpha} \\ \vdots \\ \frac{\sum_{j\in\mathscr{J}\setminus\mathscr{J}_f} s_j}{1-\sum_{j\in\mathscr{J}_f} s_j}\omega_j^f - \omega_{J_f}^f + \frac{1}{1-\sum_{j\in\mathscr{J}_f} s_j}\frac{\sigma}{\alpha}\end{pmatrix}$$

and

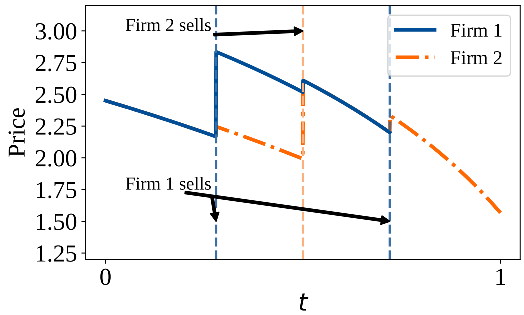
$$D_{\mathbf{p}}g^f(\mathbf{p}) = \left(\left(\begin{array}{c} \frac{s_k}{1-\sum\limits_{j\in\mathcal{J}_f} s_j} \\ \vdots \\ \frac{s_k}{1-\sum\limits_{j\in\mathcal{J}_f} s_j} \end{array} \right)_{k\in\mathcal{J}_f}, \left(\begin{array}{c} \frac{\alpha}{\sigma} \frac{-(1-\sum\limits_{j\in\mathcal{J}_f} s_j)(1-\sum\limits_{j\notin\mathcal{J}_f} s_j)+(\sum\limits_{j\in\mathcal{J}_f} s_j)(\sum\limits_{j\notin\mathcal{J}_f} s_j)}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \omega_j^f + s_k \frac{\sum\limits_{j\in\mathcal{J}_f} s_j}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \\ \vdots \\ \frac{\alpha}{\sigma} \frac{-(1-\sum\limits_{j\in\mathcal{J}_f} s_j)(1-\sum\limits_{j\notin\mathcal{J}_f} s_j)+(\sum\limits_{j\in\mathcal{J}_f} s_j)(\sum\limits_{j\notin\mathcal{J}_f} s_j)}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \omega_j^f + s_k \frac{\sum\limits_{j\in\mathcal{J}_f} s_j}{(1-\sum\limits_{j\in\mathcal{J}_f} s_j)^2} \end{array} \right)_{k\notin\mathcal{J}_f} \right)$$

For large σ the term in front of ω_j^f vanishes relative to the probability.

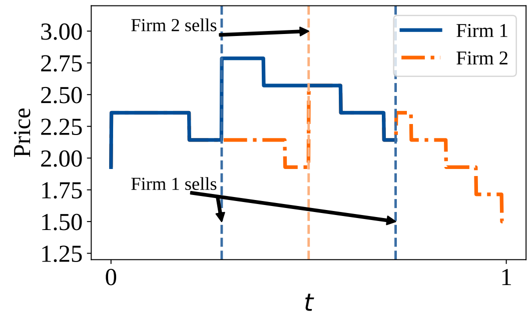
D Additional Empirical Results

Figure 15: Price Path Realizations comparing Benchmark model to Heuristics

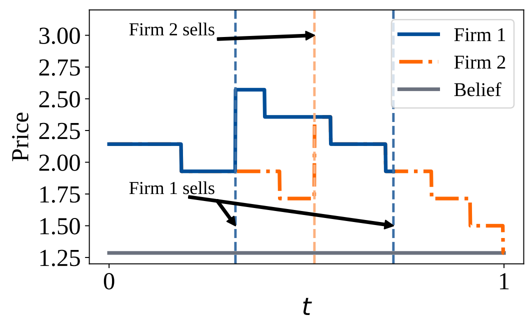
(a) Price paths for the benchmark model



(b) Price paths for the lagged algorithm

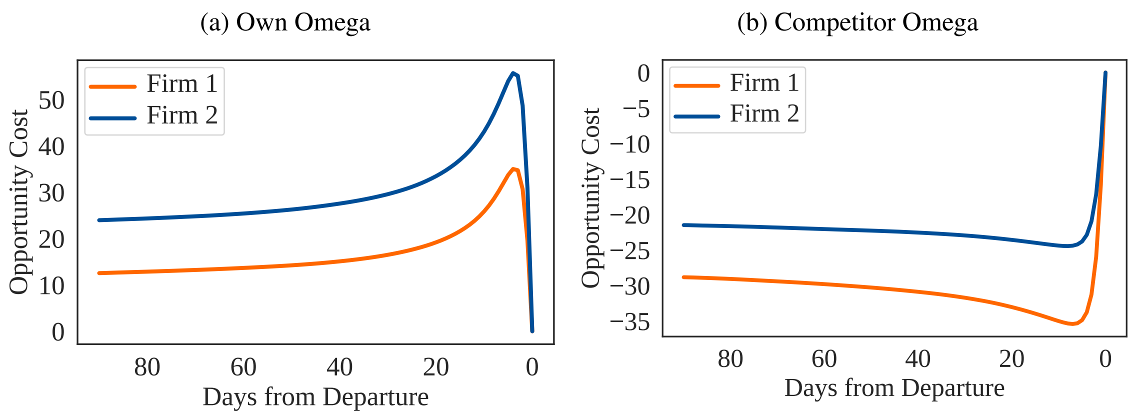


(c) Price paths for the deterministic algorithm



Notes: We assume demand follows a logit specification with an initial capacity vector of $\mathbf{K}_0 = (2,2)$. Time is continuous for $t \in [0,1]$. There are three panels: panel (a) depicts the equilibrium price path for the benchmark model, panel (b) considers prices if firms use the lagged model, and panel (c) considers prices if firms use the deterministic model. The vertical lines mark realized sales times; the color denotes the firm that received the sale. These simulations correspond to the parameter values $\delta_j = 1$, $\alpha = 1$, $\rho = 1$, $\lambda = 10$ and $\mathbf{K}_0 = [2,2]$. In the heuristic model, firms assume that the competitor prices at the level given by the grey line.

Figure 16: Recreation of Fig. 11 with restricted initial capacity



Note: Panel (a) reports the own-firm opportunity cost over time for both firms. Panel (b) reports the cross-firm competitor opportunity cost over time for both firms.

Table 5: Recreation of Table 3 with restricted initial capacity

	Price	Firm 1 Rev.	Firm 2 Rev.	CS	Welfare	Q	LF	Sellouts
Benchmark	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
– own ω only	97.4	99.1	99.0	101.2	100.3	101.7	100.6	107.7
– no ω	86.5	91.1	86.2	98.7	94.4	103.4	101.3	259.5
Deterministic	94.1	95.5	95.9	108.4	103.0	105.1	102.0	178.3
Lagged	102.0	100.3	101.2	104.4	102.9	100.6	100.2	104.0
Uniform	97.5	78.1	77.3	113.7	98.5	101.1	99.9	242.0

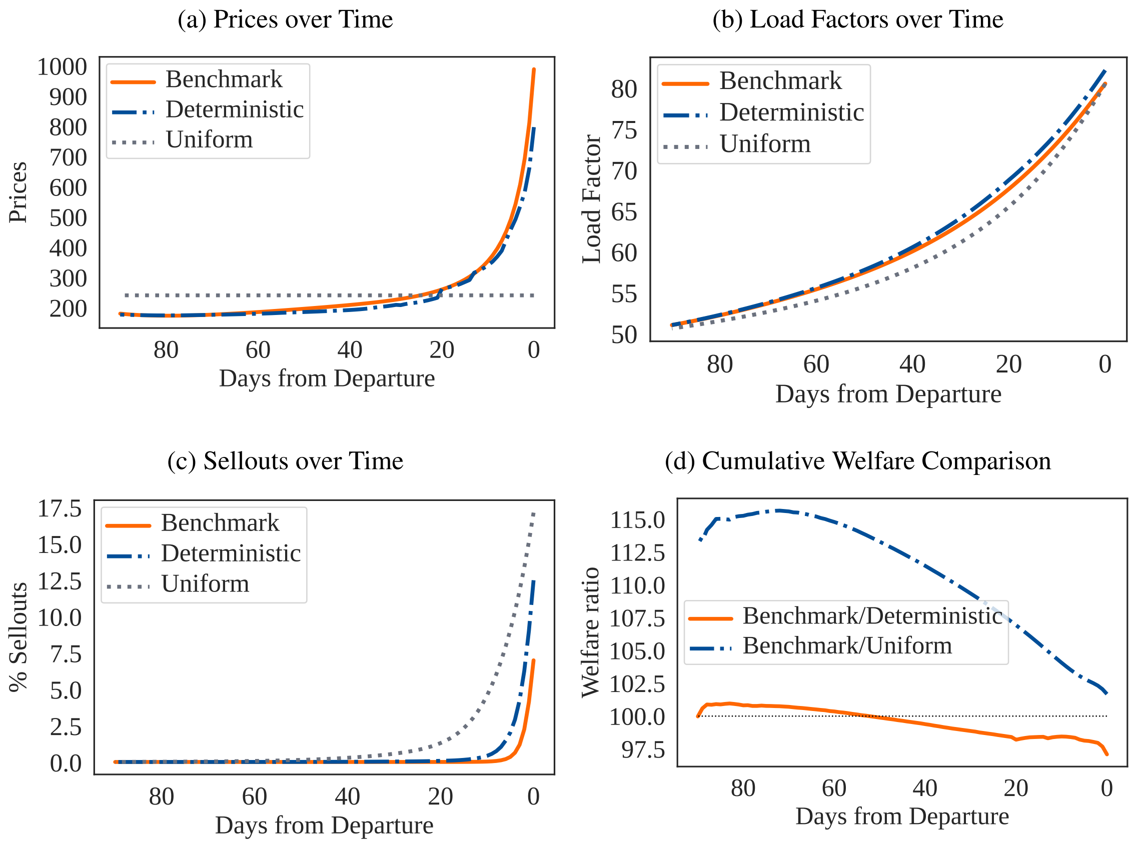
Note:

Table 6: Firm Profits Across Counterfactuals

Firm 1 Preference	Firm 2 Preference	Fraction of Markets (r, d)
Benchmark	Benchmark	57.0%
Deterministic	Deterministic	23.1%
Deterministic	Benchmark	6.2%
Benchmark	Deterministic	13.8%

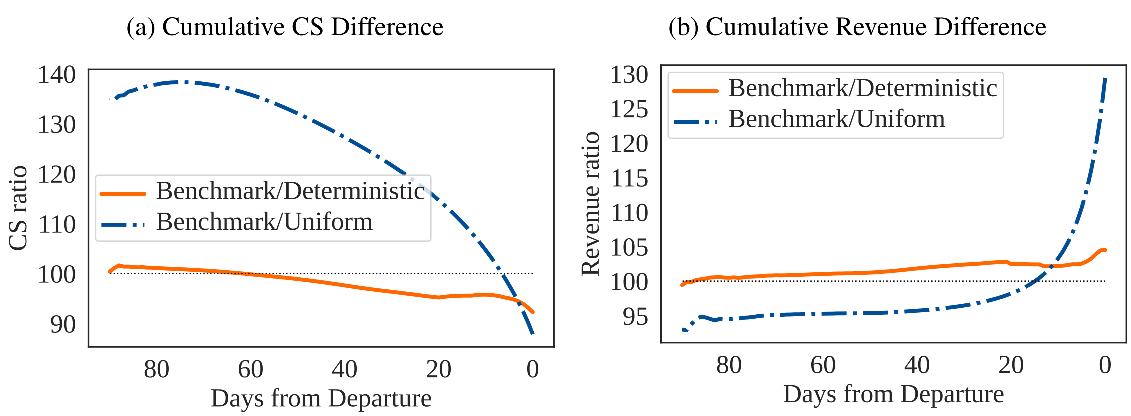
Note:

Figure 17: Recreation of Fig. 12 with restricted initial capacity



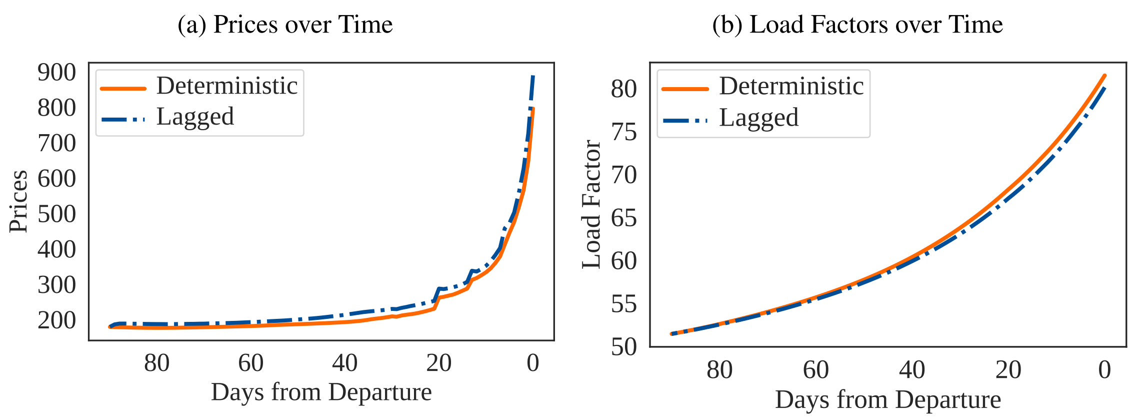
Note:

Figure 18: Recreation of Fig. 13 with restricted initial capacity



Note: Panel (a) reports the own-firm scarcity effect over time for both firms. Panel (b) reports the cross-firm competitor scarcity effect over time for both firms.

Figure 19: Recreation of Fig. 14 with restricted initial capacity



Note: