Homophily and Influence: Pricing to Harness Word-of-Mouth on Social Networks

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Abstract

Large-scale social platforms have enabled marketers to obtain rich data on the structure of word-of-mouth (WOM) networks and the correlation of friends’ preferences (network assortativity). However, the literature is silent on how the similarity or difference of friends’ reservation prices for a product should affect the optimal price and advertising levels for that product. To answer this question, we present an analytical model of informative advertising and pricing over a social network. Connections between consumers are added in a way that allows neighbors’ preferences to be positively or negatively correlated, thereby introducing homophily or heterophily in the model. Consumers may learn about products either directly via advertising, or via WOM spread by their peers who have adopted a product. We find that in the typical scenario when blanket advertising is not affordable, firms set a price lower than the naïve optimum in order to leverage the social value of more price-sensitive customers. We also characterize the relationship between assortativity and the marketing instruments (price and advertising) of the firm, linking the comparative statics to the achieved penetration of the available market. Finally, we show how this relationship may change under intense competition for a market of high-valuation customers.

Keywords: informative advertising, pricing, word-of-mouth, assortativity, homophily, social influence.
1 Introduction and Motivation

The advertising revenues of social platforms (SPs) have reached unprecedented heights during the current decade. In the US alone, Facebook collected about $4 Billion in 2014 from ads displayed on their site or in their mobile app. Across the industry, US ad revenues now total to about $6 Billion a year (Freid 2014). The trends are similar around the globe: in 2014, Chinese tech giant Tencent generated about $1.5 Billion in ad revenues, an increasing fraction coming from ads placed on their social networking and messaging platforms, e.g. Qzone and WeChat (Weixin). Also in China, the 2014 advertising revenues of Sina’s Weibo were about $250 Million, a number that is growing at a 50% annual rate.\footnote{See http://www.tencent.com/en-us/content/at/2014/attachments/20141112.pdf (accessed December 15, 2014) and http://usa.chinadaily.com.cn/china/2014-11/15/content_18921569.htm (accessed December 18, 2014)}

A common advantage of SPs over other advertising media lies in that beyond reaching a vast number of consumers – the two largest social networks, Facebook and Qzone post about 2 Billion monthly active users combined (Bischoff 2014) – they can also identify and reach those consumers’ peers (Ellison and Boyd 2007). Most SPs provide sophisticated communication interfaces that enable rich interaction between their users. For instance, Facebook, WhatsApp, WeChat, Google+, and Twitter all allow users to share and interact with both user-generated and firm-published media content of various forms such as text, pictures, audio or video messages. In addition, services such as WeChat’s “red envelope” for monetary gift-giving during Chinese New Year's also allow SPs and advertisers a glimpse at the strength of the underlying social connections between users. Importantly, observing user interactions with each other’s content allows the platforms to map the influential communication paths within their user base. Put together, these paths constitute the word-of-mouth (WOM) network on the platform.

Marketers and SPs themselves have been constantly looking to improve their ways of monetizing such WOM networks. Many SP services provide marketer-sponsored profiles, or “brand
pages,” where companies gather followers and fans to communicate regular updates to them. As of December 2014, Katy Perry, Barack Obama, and YouTube have 62 million, 52 million, and 47 million followers on Twitter, respectively. The marketing firm Izea reported that a Tweet by Kim Kardashian could cost as much as twenty thousand U.S. dollars (Kornowski 2013). The original business model of Pinterest, an SP that allows users to organize visual bookmarks and share this with their peers, was built on monetizing commercial product-laden user content posted on the site – a casual user’s “pin” was reported to generate as much as 78 cents in sales. Sina Weibo and Tencent have launched similar services to promote “Key Opinion Leaders” (KOL) to influence users’ peers. Finally, sites such as Facebook and LinkedIn have introduced the concept of “social advertising,” wherein marketers may use consumers’ social information to select their advertising targets (Bakshy et al. 2012, Tucker 2012). These efforts did not go unrewarded: whereas the cost per impression of advertising on Facebook has increased by about 600% over the past year, the ROI of Facebook advertising has nevertheless doubled over the same period.

Interestingly, most of marketers’ efforts are oblivious to the price of the advertised product. This is problematic since for most products, WOM is more likely to be spread by consumers who have purchased and therefore experienced the product (e.g., mobile apps that automatically share WOM information with all peers of the customer). In such scenarios, the influence process on the WOM network can break down at consumers whose reservation price for the product is too low. Considering the marketers’ perspective, raising price not only trades off the number of interested buyers for higher margins, but also excludes a bulk of peer-to-peer relationships that could otherwise increase the demand of the firm via WOM. In addition, the magnitude of this effect depends on the correlation of network neighbors’ preferences, or the assortativity of the network. In networks with higher assortativity, high-valuation consumers have more high-valuation friends.

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3Part of the CPM increase may correspond to format changes implemented by Facebook during the last 12 months. The full report is available at http://kenshoo.com/social-marketers-boost-roi-holiday-season-facebook-advertising-continues-perform/ (accessed December 10, 2014).
making them more likely to learn about the advertised product via WOM.

It is widely known that in most social networks, connected users tend to exhibit positively correlated patterns, or homophily,⁴ of demographics, psychographics, and product preferences (McPherson et al. 2001, Ugander et al. 2011). Empirically separating homophily from social influence has been a central question in the empirical WOM literature (Aral et al. 2009, Durlauf and Ioannides 2010, Iyengar et al. 2011). Still, the tools offered by the top SPs aimed at monetizing WOM do not distinguish between this correlation and social influence. Rather, they implicitly assume that the WOM links observed as frequent interactions between connected consumers indicate that these consumers also share their values and beliefs (Klaassen 2008). However, this is clearly not always the case – for instance, recent empirical work has discovered considerable variation in the assortativity of social networks (Mislove et al. 2007, Newman 2002, Rivera et al. 2010). In sum, the current industry best practices of ignoring the assortativity of the WOM network and setting advertising independently from the product’s price may lead to suboptimal profits to marketers.

The extant literature on advertising to networks of consumers (Domingos and Richardson 2001, Galeotti and Goyal 2009, Kempe et al. 2003) paints a picture consistent with industry practice. First, it is commonly accepted that blanket advertising is wasteful and that leveraging the WOM network to spread product information can be beneficial to marketers (Galeotti and Goyal 2009, Zubcsek and Sarvary 2011). Second, most papers focus solely on optimizing the advertising decision of firms and consider that prices can be set independently of the subsequently chosen advertising strategies. The assumption allowing this separation is that WOM behavior is treated as independent from consumers’ reservation price - in particular, everyone in the market is assumed to spread WOM according to their connectivity, irrespective of their valuation for the advertised product (Galeotti and Goyal 2009). While this simplification indeed renders the assortativity of

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⁴In the literature, both homophily (heterophily) and assortativity (dissortativity) are used to mean the positive (negative) correlation between network neighbors’ attributes. Throughout our work, we adopt assortativity to denote the correlation (of any sign) between neighbors’ preferences, and homophily (heterophily) to mean an explicitly positive (negative) correlation.
the WOM network irrelevant, assuming that consumers with low valuations for the product also spread WOM leads to overestimating WOM and under-advertising by firms. Given that the literature commonly embraces a notion that goes opposite to this assumption (Berger and Schwartz 2011, Biyalogorsky et al. 2001, Rust et al. 1995), the lack of guidance on whether and how the (assortativity of the) WOM network may affect the optimal price for a product advertised on a SP is an important gap in the literature.

Should the structure of WOM communication in the market inform the price charged by firms? How should firms adjust their advertising strategies based on the degree of assortativity among the consumers targeted? To answer these questions, we consider an analytical model of informative advertising that adds network connections between consumers in a circular market such that the probability that two consumers are connected depends on the similarity of their preferences. Our setup extends the analytical model of pricing and advertising considered by Galeotti and Goyal (2009) in two important ways. First, we allow neighbors’ preferences to be positively (negatively) correlated, thereby allowing for the WOM networks in our model to exhibit homophily (heterophily). Second, we no longer assume that each consumer, including those consumers who do not receive positive utility from purchasing the advertised product, may spread WOM. Instead, firms may only rely on WOM spread by those consumers who both have been reached by advertising and have a reservation price above the advertised price of the product.

We derive optimal strategies both for a monopolist marketer and for two horizontally differentiated competing firms. We obtain that unless advertising is “essentially free”, the monopolist finds it optimal to charge a price lower than that under full information. The intuition behind this result is that by lowering the price, the firm earns more WOM that leads to ultimately informing more high-valuation consumers. Exploring how the optimal strategy changes with the degree of assortativity in the network, we find that both price and advertising may respond non-monotonically as the correlation of friends’ preferences increases. In particular, we demonstrate examples wherein,
*ceteris paribus*, the increase of assortativity may trigger a U-shaped pattern of price or an inverse U-shaped pattern of advertising. We also show that the overall strength of WOM increases advertising only when consumers’ valuation for the monopolist’s product is high relative to the cost of advertising. Extending this result, we find that the strength of WOM interacts with assortativity: As WOM grows stronger in a homophilous market, the monopolist may advertise more if the new connections decrease the degree of assortativity in the network. Finally, we find that the impact of the network structure of WOM is different under competition. While firms in a duopoly typically set higher prices and lower advertising levels than the monopolist, in heterophilous markets these patterns may reverse. Specifically, in markets where advertising is expensive and WOM is weak, if the WOM network exhibits heterophily then competing firms may set lower prices than their monopolist counterparts. On the other hand, in markets where advertising is cheap and WOM is strong, the heterophily of the network leads competing firms to advertise more than a monopolist.

Our findings have several important implications. First and foremost, we show how accurate measures of preference assortativity in markets can help firms calibrate both their price and advertising levels to maximize the return on advertising campaigns. Second, studying a broader class of assortativity patterns than extant analytical models, we show how treating WOM behavior as independent from consumers’ reservation price yields upward-biased price recommendations. Finally, we derive insights for the owners of networked advertising platforms (e.g., Facebook), identifying circumstances in which the most profitable strategy to grow the network may be to create relationships between “birds of different feathers.”

The rest of our paper is organized as follows. Next, we review the related literature. Section 3 presents our model. In section 4 we detail our analysis methods and present our findings for a monopolist. Section 5 extends the model to a duopoly, and section 6 concludes the paper and discusses the implications of our findings. For easier exposition, we have relegated the analytical proofs to Appendix A.
2 Related Literature

Our work extends the literature on informative advertising and the literature on WOM. We study a one-shot game in a duopoly with horizontally differentiated products (Grossman and Shapiro 1984, Soberman 2004) where firms use advertising to inform consumers about the existence and the price of their products. We make the simplest assumptions about the marketing instruments, ignoring the possibility of targeting advertising (Iyer et al. 2005) or nonuniform pricing (Chen and Iyer 2002). On the other hand, we do introduce informational network externalities to our model: Consumers may let their peers know about the advertised product via WOM similar to the mechanism considered in Galeotti and Goyal (2009) and Zubcsek and Sarvary (2011). However, we follow the approach of Campbell (2013) and Ajorlou et al. (2014) and refine the model of WOM communication by restricting it to those consumers whose reservation price is low enough. Finally, we assume away any utility that consumers may derive from sharing information (Campbell et al. 2013), and also ignore the possibility of any payoff externalities (e.g., Amaldoss and Jain (2005), Zhang and Sarvary (2014)). Our primary objective is to study the role of assortativity in the WOM process, and characterize how it shapes firms’ optimal pricing and advertising strategies.

Since the seminal paper by Katz and Lazarsfeld (1955) introduced a two-stage model wherein "opinion leaders" affect the mass market through social influence, WOM has been a central topic in the marketing literature (Bass 1969, Brooks 1957, Godes and Mayzlin 2004). While much of the work studying WOM processes focused on behavioral factors affecting information transfer (Berger and Schwartz 2011, Brown and Reingen 1987, Chevalier and Mayzlin 2006, Herr et al. 1991), the rapid development of communication technologies in the new millennium has boosted empirical WOM research that is primarily concerned with network effects of social influence (Domingos and Richardson 2001, Goldenberg et al. 2009, Katona et al. 2011, Stephen and Toubia 2010, Van den Bulte and Lilien 2001, Watts and Dodds 2007).
It is well known that in SPs that provide the domain for many of the WOM processes documented in the literature, connected individuals tend to be similar to each other (Bakshy et al. 2012, Lewis et al. 2012, Ugander et al. 2011). However, the exact degree of assortativity varies by SP (Mislove et al. 2007, Newman 2002, Rivera et al. 2010) and even within the same network, preference assortativity may vary across product categories (Reingen et al. 1984). This underlines the importance of considering assortativity in any marketing strategy that relies on WOM networks. Marketers realized the importance of this phenomenon at least as far back as Reingen et al. (1984), who concluded that “brand congruence” can vary greatly across product categories such as television shows, restaurants, detergent, and personal products. In the same vein, a more recent study by Hill et al. (2013) found that the effectiveness of predicting brand engagement on Twitter using social network data varied by product type, suggesting the importance of assessing product-specific homophily in designing WOM campaigns. In light of these findings, it’s interesting that network researchers have dedicated the most attention to homophily in the context of separating it from social influence using historical data (Aral and Walker 2012, Durlauf and Ioannides 2010).

Contrastingly, normative research incorporating the assortativity of consumer WOM networks into firm decisions – in no small part due to the complexity of the problem area – is virtually absent. Indeed, analytical models of product markets with social influence such as Campbell (2012), Galeotti and Goyal (2009), Katona (2013), Zubcsek and Sarvary (2011) have been primarily concerned with the degree distribution in the network of social interactions. Thus, the ironic contrast that these two streams of literature create is as follows: On the one hand, we know that in networks of social influence, the (product) preferences of network neighbors tend to be correlated. On the other hand, the degree of assortativity is not even considered by most analytical models built to characterize optimal firm behavior over a network of consumers.

There are a few notable exceptions attempting to abridge the abovementioned gap. Galeotti and

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5In most cases, this separation is difficult due to the reflection problem (Aral et al. 2009, Manski 1993).
Mattozzi (2011) study the candidate positioning and advertising decisions of two competing political parties in the presence of homophily among voters. They show that in voting bases with higher homophily, the parties are more likely to select extremist candidates and voter misperception (of candidates’ position) is higher, ultimately leading to more polarized policy outcomes. Haenlein and Libai (2013) consider the (targeted) advertising decision of a monopolist when the underlying network of consumers exhibits homophily in customer profitability to the firm, concluding that when “revenue leaders” (i.e., customers with high profitability) tend to be densely connected, then targeting revenue leaders is likely to outperform targeting opinion leaders. Chuhay (2012) considers the product design, pricing and advertising decisions of a firm targeting a networked market of two types of consumers. Related to the questions we study, he finds that for sufficiently high levels of network assortativity, the firm manufactures a product of the best possible quality, and the price charged decreases with the correlation of neighbors’ preferences. Finally, Campbell (2013) splits a continuum of users (differing in their valuations for a monopolist’s product) into a high-valuation and a low-valuation group and defines the probability that two consumers are connected based on their membership in these groups. Analyzing the optimal informative advertising and pricing of a monopolist under such circumstances, he finds that for very high levels of homophily, the monopolist may set a price higher than it would under full information and no assortativity, effectively pricing the low-valuation consumers out of the market.

Just like these papers, we take the degree of assortativity in the network of consumers as given\(^6\) and study its impact on optimal firm behavior. Our contribution above earlier work is (i) considering simultaneous advertising and pricing decisions (ii) in a market with a continuum of consumer types and a naturally complex domain of network assortativity, while (iii) also accounting for competition. Our model is presented next.

3 Model

We use the standard circular city model of Salop (1979) where consumers are uniformly distributed around a market with a circumference of 1. The risk-neutral monopolist firm may be located at any point in the market, and consumer \( i \)'s utility from buying and consuming the product from the firm is

\[
U_i = V - p - t \cdot d_i,
\]

where \( V > 0 \) is the benefit that a consumer receives from a product that perfectly suits her taste, \( p \) is the price charged by the firm, \( t > V \) is the importance of personal taste and \( d_i \) is the distance on the circle between consumer \( i \) and the firm. Without loss of generality, we assume that the marginal cost of the firm is 0, so it makes \( p \) units of marginal surplus on its sales.

The monopolist may inform consumers about their products via advertising or word-of-mouth (WOM). We assume that the firm cannot target its advertising, only choose its intensity \( \alpha \geq 0 \). When the firm chooses advertising intensity \( \alpha \), any consumer will find out about its product via advertising with probability \( \alpha \), while the firm will incur \( c \cdot \alpha^2 \) advertising cost for some \( c > 0 \).

Following Galeotti and Goyal (2009), we model social interactions so that individuals are located in a social network that can be complex and take on a variety of forms. For any consumer \( i \), the level of social interaction is characterized by a number \( d(i) = m \), the degree of consumer \( i \). We assume that each individual draws \( m \) others (their neighbors) with probability \( P(m) \geq 0, m \in \{1, 2, \ldots, \bar{m}\} = M \), and \( \sum_{m \in M} P(m) = 1 \).

In Galeotti and Goyal (2009), neighbors were drawn uniformly at random from the entire circle of consumers. Their model thereby assumed complete independence between the product preferences of related actors. To study the role of assortativity in the network, we allow consumers to pick their peers from a more general (atomless) neighbor distribution \( F : [0, 1] \to [0, 1] \). However, we assume that the extent of assortativity is the same at any position around the circle by restricting
the corresponding neighbor density function \( f : [0, 1] \rightarrow \mathbb{R} \) so that consumer \( v \) picks consumer \( w \) with relative likelihood \( f(d_{vw}) \). For notational convenience, we further let \( f(x) = f(1 - x) \) for any \( x \in [0, 0.5] \). Finally, since we are interested in the role of assortativity in the network, we assume that \( f \) is differentiable on \((0, 0.5)\) and that for any \( x \in (0, 0.5) \), one of the following cases holds:

- **Case 1** (Homophily): \( \partial f(x) / \partial x < 0 \).
- **Case 2** (No assortativity): \( f(x) \equiv 1 \).
- **Case 3** (Heterophily): \( \partial f(x) / \partial x > 0 \).

Thus, in the homophily case, friends are more likely to be located close to each other on the circle (i.e., their preferences are positively correlated) and in the case of heterophily, the opposite is true. Finally, in the no assortativity case, friends’ preferences are completely independent.

We note that our model does not imply that when a person \( i \) draws \( j \), then \( j \) draws \( i \) as well. Therefore, the network of social influence is directed, and degree here may refer to the number of people an individual gets information from or the number of people an individual is influenced by (Galeotti and Goyal 2009). However, as the draws of everyone exhibit circular symmetry, everyone is, on average, drawn by the same number of people. Since this implies that every individual is (on average) equally influential, for expositional simplicity, we use *degree* to denote both in-degree and out-degree in the remainder of the paper. Further, to illustrate our contribution through a simpler analysis, we assume that the network is *regular*, i.e. that each consumer at location \( x \) has the same degree \( d(x) = d \) for some \( d \in \mathbb{N}^+ \).

WOM communication is then modeled as follows. Any consumer may learn about the product directly via advertising or indirectly via WOM from any of their neighbors who received advertis-

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7Our analytical results are straightforward to translate to the general network structure characterized by \( M \) and \( \{ P(m) \}, m \in M \). Moreover, using the analysis methods of Kempe et al. (2003), it can be shown that optimal pricing and advertising under any probabilistic multi-step “cascade model” of WOM arises as a convex combination of solutions on regular networks under the WOM model of Galeotti and Goyal (2009).
ing. Like earlier models (Galeotti and Goyal (2009), Zubcsek and Sarvary (2011)), however, we also limit the reach of WOM to one step. 8 If consumer \( i \) is targeted with an advertising message, then \( i \) learns about the marketer’s product and its price. If \( i \) also prefers to buy the product (over the outside option of not making a purchase at all), then they will communicate the product information (including the price) to all their neighbors. However, if the marketed product is not \( i \)’s most preferred choice then no WOM will be spread (Campbell 2013). This is an important difference from the model of Galeotti and Goyal (2009) as in our model, not only the demand but also the size of the population actively spreading information does depend on the prices that the firms set.

4 Analysis – Monopoly

In this section, we discuss the optimal behavior of the monopolist. For convenience, we assume that the firm is at location 0 and number the other locations in the market from -0.5 to 0.5, with the positive numbers being clockwise from the firm. Without loss of generality, we assume that the firm sets a price \( p \) such that the consumer at 0.5 (weakly) prefers to stay out of the market. This is equivalent to \( V - 0.5 \cdot t - p \leq 0 \Leftrightarrow p \geq V - t/2 \). Let \( z \) be the distance of those consumers from the firm who are indifferent between buying and not buying the product, i.e. \( z = (V - p)/t \).

Before discussing the monopolist’s optimal strategy according to the model presented in section 3, we enumerate over two benchmark strategies derived for models introduced in prior work. These are presented next.

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8By doing so, we turn away from studying diffusion dynamics and restrict our attention to how the structure of the WOM network impacts firms’ optimal strategies by affecting the reach of advertising. However, we note that since the advertising messages are assumed to contain the price of the offering, our formulation is essentially equivalent to assuming no discounting.
4.1 Benchmark Strategies Derived from Prior Models

4.1.1 Ignoring Word-of-Mouth

The simplest strategy of the firm ignores the presence of WOM. The results are trivial to derive from the first-order conditions of the model: The monopolist sets $p_0 = V/2$ and

$$\alpha_0 = \begin{cases} V^2/4tc & \text{if } V^2 < 4tc, \\ 1 & \text{otherwise (blanket advertising)}. \end{cases}$$

4.1.2 Ignoring How Price Reduces WOM

In this scenario, the firm is assuming that all connections and consumers take part in the WOM process. (This is too optimistic because it overlooks that the consumers priced out of the market would not spread the message about the product.) Profit-maximizing behavior can be computed just like in Galeotti and Goyal (2009): The monopolist optimizes

$$\Pi_{AT} = p \cdot \left\{ \frac{2(V - p)}{t} \cdot \left[ 1 - (1 - \alpha)^{d+1} \right] \right\} - c\alpha^2,$$

where $\Pi_{AT}$ is the profit corresponding to the average consumer in the network under the firm’s assumption. In optimum, $p_{AT} = V/2$ and $\alpha_{AT}$ is the positive solution to

$$\frac{(1 - \alpha)^d}{\alpha} = \frac{4ct}{(d + 1) \cdot V^2}.$$ 

We note that in this model, the monopolist never opts for blanket advertising.

4.1.3 Discussion

The above results show that when price is assumed to have no impact on any consumer’s WOM behavior then even after such a strikingly different treatment of WOM (no-one talks versus everyone informed does), the pricing decision of the firm becomes independent from the subsequent advertising decision. Moreover, the price essentially simplifies into an expression that is also independent from the structure of the underlying social network. However, when consumers whose
reservation price is too low do not spread WOM, such strategies are overly naïve as we explain below.

According to our notation, the available market to the firm is the market of size \(2z\) from \(-z\) to \(z\). To see how price affects WOM, consider the sets \(I = [-z, z]\) and \(O = [-0.5, -z) \cup (z, 0.5]\). In the network of social influence, \((I, O)\) is then a cut, or a partitioning of the nodes in the network. Furthermore, since all the customers in \(O\) strictly prefer the outside option to the monopolist’s product, it is easy to see that edges with at least one end in \(O\) cannot contribute to firm profits: useful WOM may only happen along edges connecting customers in \(I\) to each other (Figure 1).

The strategies discussed in section 4.1 assume otherwise and are therefore bound to produce suboptimal results. In the next section, we present a method that correctly accounts for the impact of consumers’ reservation price on their WOM behavior.

### 4.2 Accounting for Reservation Price Effects

In this section, we derive the strategy that incorporates the appropriate model of social influence in the advertising and pricing of the firm. Note that for the monopolist, the problem of finding the optimal \((\alpha, p), \alpha \in [0, 1], p \in \max\{0, V - t/2\}, \infty\) strategy can be phrased as that of finding the optimal \((\alpha, z), \alpha \in [0, 1], z \in [0, 0.5]\) strategy. In our analysis, we follow the latter approach.

Let the advertising level chosen by the firm be \(\alpha\). The expected quantity demanded by consumer \(x\) with \(x < z\) arises from two components: the direct impact of advertising and the likelihood of WOM influence when advertising does not reach the consumer at \(x\). The prior term is simply \(\alpha\), while the second term is the product of \((1 - \alpha)\) and the probability that each friend of \(x\) did either not receive advertising or finds the price \(p\) too high. For one such friend, Figure 2 displays how to calculate the probability of the complementary event which is \(\alpha \cdot [F(z + x) + F(z - x)]\). Since the
network is \( d \)-regular, the demand for the monopolist’s product is then

\[
Q(\alpha, z) = \alpha \cdot 2z + (1 - \alpha) \cdot \int_{-z}^{z} 1 - \{1 - \alpha \cdot [F(z - x) + F(z + x)]\}^d dx. \tag{1}
\]

Let \( G(z, x) = F(z - x) + F(z + x) \) for \( z, x \in [0, 0.5] \). Then the profits to the monopolist are

\[
\Pi(\alpha, z) = p \cdot Q(\alpha, z) - c \cdot \alpha^2 \tag{2}
= (V - tz) \cdot \left\{ \alpha \cdot 2z + 2 \cdot (1 - \alpha) \cdot \int_{0}^{z} 1 - [1 - \alpha \cdot G(z, x)]^d dx \right\} - c \alpha^2. \tag{3}
\]

Thus, the monopolist’s optimization problem is then setting

\[
(\alpha^*, z^*) = \arg \max_{\alpha \in [0, 1], z \in [0, 0.5]} -c \alpha^2 + 2(V - tz) \cdot \left\{ \alpha z + (1 - \alpha) \cdot \int_{0}^{z} 1 - [1 - \alpha \cdot G(z, x)]^d dx \right\} \tag{4}
= \arg \max_{\alpha \in [0, 1], z \in [0, 0.5]} -c \alpha^2 + 2(V - tz) \cdot \left\{ z - (1 - \alpha) \cdot \int_{0}^{z} [1 - \alpha \cdot G(z, x)]^d dx \right\}, \tag{5}
\]
given the parameters \( V > 0, t > 0, c > 0, F : [0, 1] \to [0, 1], d \in \mathbb{N}^+ \) and the constraint \( V - tz > 0 \).

### 4.3 Pricing to Harness Word-of-Mouth

In section 4.1.3, we established that reservation price effects may impact the WOM process that unfolds after the advertising campaign. This is a major change in perspective compared to earlier work wherein the advertising choice perfectly determined WOM independently of the price of the product. In our model, the marginal consumer at location \( z \) may have social ties to otherwise unreached consumers (unless the monopolist advertises to everyone in the market). This presents a \textit{WOM opportunity} that the firm can capitalize on by lowering its price. Formally,

**Lemma 1.** \textit{Holding advertising constant at some} \( 0 < \alpha < 1 \), \textit{lowering price increases not only the size of the available market, but also its (percentage) penetration by the monopolist.}
Lemma 1 captures the quintessential structural difference between our work and the prior models discussed in section 4.1, highlighting how price impacts WOM. It has powerful implications on the monopolist’s optimal pricing, as captured in the following proposition.

**Proposition 1.** For the optimal monopoly price \( p \), we have

\[
\max(0, V - t/2) \leq p \leq V/2.
\]

Further, if

\[
c > \frac{V}{2} \cdot \int_{0}^{\frac{V}{2t}} \left[ 1 - G \left( \frac{V}{2t}, x \right) \right]^d dx
\]

then \( p < V/2 \).

**Proof:** See Appendix A.

Proposition 1 says that when blanket advertising is suboptimal then the monopolist must lower the price from the naïve optimum of \( V/2 \) to maximize profits. The intuition is that when \( \alpha = 1 \) is optimal then every consumer is informed and so \( p = V/2 \) is the optimal price just as in the models discussed in section 4.1. However, for \( 0 < \alpha < 1 \) (the left side of which is obvious), Lemma 1 implies the following two things. First, the optimal price is at most \( V/2 \) – in particular, any price which is too high in the model of Galeotti and Goyal (2009) cannot be the optimal price in our model. Second, the derivative of the profit function in \( z \) at \( z = V/2t \) is positive. This means that the monopolist prefers a price lower than \( p = V/2 \) in this case.

Whereas Proposition 1 focuses on the threshold cost of advertising above which the monopolist charges a lower price than the naïve \( p = V/2 \), it actually characterizes the pricing decision in terms of the interplay of all five parameters \( V, t, c, d, \) and \( F(\cdot) \) (the latter implicit in the function \( G \)). In fact, for most parameter combinations, the advertising cost threshold is very low – for the broad range of examples analyzed in the next section, we always found blanket advertising to be wasteful, implying \( p < V/2 \) for the optimal pricing strategy of the firm.
4.3.1 The Role of Brands as Drivers of Assortativity

It is interesting to note that Proposition 1 seems to contradict the results of Campbell (2013) on assortativity. In particular, while Campbell (2013) does show that $p < V/2$ for networks with no assortativity, he also constructs examples with strong homophily in the network where the monopolist finds it optimal to charge a price above the naïve optimum. In this section, we show how the difference between our and Campbell’s (2013) results is driven by the assumed structure of assortativity, and explain what this means to marketers.

To model assortativity in the network, both Campbell (2013) and Chuhay (2012) build on the work of Newman (2002), and model consumer types $H$ (for all $|x| \leq s, s \in [0, 0.5]$) and $L$ (for all $s < |x| \leq 0.5$), characterizing the network by a $2 \times 2$ matrix $E$, with $e_{ij}$ corresponding to the relative fraction of (directed) edges in the network between type $i$ and type $j$ consumers. To study undirected networks, Campbell (2013) further adds the restriction that $e_{12} = e_{21}$. While our networks are not undirected in the strict sense, following the discussion in section 3, it is most natural that we also include this assumption. We can then further simplify our model so that the probability that a high-type individual draws another high-type as friend is $0 \leq e \leq 1$. Then

$$E = \begin{bmatrix} 2se & 2s(1-e) \\ 2s(1-e) & 1-2s(2-e) \end{bmatrix},$$

and the probability that a low-type consumer draws a high type as friend is $2s(1-e)/(1-2s)$.

Next, we show that under this model there exist situations in which the monopolist charges a price $p > V/2$, and demonstrate the prevalence of such parameter combinations.

**Proposition 2.** For any $V > 0, t > V, 0 < d \in \mathbb{Z}, 0 < c$, there is an assortativity structure defined by $(e, s)$ at which the monopolist charges $p > V/2$ in the two-type model of assortativity.

**Proof:** See Appendix A. \hfill \Box

We are adopting some changes to Campbell’s (2013) notation for convenience. Further, to allow for a comparison with our basic model, we keep our assumption that WOM travels only one step in the network.
Proposition 2 states that when the assortativity of the network is driven by the interaction (or separation) of the high and low type consumers, then it is possible that the monopolist prices above the naïve optimum. The proof of the proposition reveals that such high prices tend to be optimal when there is a (near) complete separation between the two types, i.e., when high types only connect to high types, and low types to low types.

[– Insert Figure 3 around here –]

[– Insert Table 1 around here –]

To make sense of the apparent contradiction between Propositions 1 and 2, it is important to explain how the two models of assortativity differ, and how those differences may drive different pricing implications. Figure 3 shows the location of four consumers, A, B, C, and D on the unit circle, and indicates the line at distance $s$ from the firm that separates the high and low types in the two-type model of assortativity. Table 1 looks at all six pairs of the four consumers and compares the two models of assortativity in terms of how similar they treat the two consumers in question. We note that while there are several differences owing to the fact that the two-type assortativity model dichotomizes the similarity relationship, the most striking difference is how the two models classify the similarity of consumers B and C. Since the valuations of B and C for the product are fairly similar, our model assumes that their social connections are likely also similar. However, in the model inspired by Campbell (2013), B is a high type and C is a low type consumer, and whereas their valuation for the product is similar, their different types drive them to interact with consumers of different profiles.

What could drive such ample differences between otherwise very similar consumers? The literature on homophily and network formation (Currarini et al. 2009, Jackson and Rogers 2007, McPherson et al. 2001) provides only part of the answer, leaving the question open why the distribution of consumer types should align so perfectly with the monopolist’s location. One explanation
could be that the monopolist simply sets its product design such that its location is exactly in the middle of the (then) high type segment. While this is theoretically possible, it still does not explain what drives the ex-ante structure of assortativity patterns of the network and, specifically, why that driver should be related to consumers’ valuations for the new product. We therefore turn to the more broadly applicable alternative theory of Lovett et al. (2013) proposing that brands may influence the structure of network connections in the market. This notion is consistent with the marketing literature that suggests that consumers use brands to signal their identities in social interactions (Berger and Heath 2007, Kuksov 2007).

According to the theory of Lovett et al. (2013), highly differentiated brands that are used for identity signaling are indeed likely to trigger homophilous connections between high-valuation consumers. For instance, thanks to the consistent design and platform externalities of Apple versus Samsung products, when either of those companies launches a gadget in a new product category (e.g., high-end wearable computing such as smartwatches), they will likely appeal to their own customer base, and benefit from strong brand-driven homophily that is structured like in the model of Campbell (2013). When the brand-driven homophily is very strong, it may allow the marketer to charge a price *above* the level that the naïve models reviewed in section 4.1 suggest. Interestingly, those benchmark prices already reflect the high individual brand valuations; the extra price increase is made profitable entirely by the strong homophily in the network.

On the other hand, when a largely unknown startup launches a product in an entirely new category (e.g., Audible, Inc. launching the first production-volume portable digital media player), they are more likely to face a structure of assortativity characterized in section 3 of this paper – one wherein social connections reflect the inherent similarities of consumers that are independent of their specific position on the unit circle. In this case, the marketer should act on Proposition 1, and introduce the product at a *lower* price than what the naïve models would predict based on the distribution of consumers’ valuations.
Finally, we note that Proposition 2 naturally implies that there is no high-low type distribution that is immune to brand-driven homophily:

**Corollary 1.** For $t > 0, 0 < d \in \mathbb{Z}, 0 < c, \text{ and any type distribution determined by } 0 < s < 0.5, \text{ if } e = 1 \text{ then there is a baseline valuation } V \text{ at which the monopolist charges } p > V/2 \text{ in the two-type model of assortativity.}

In other words, managers of strong brands should carefully assess the baseline valuation of the market before the launch of a new product as there is a certain range (just above $2t \cdot s$) where they should set a price higher than the one that would be optimal in the absence of WOM.

### 4.4 How Network Structure Shapes Optimal Strategies

Let us now return to our basic model formulation and consider a monopolist launching its product in two markets, A and B, simultaneously. The baseline valuation $V$, the sensitivity to product characteristics $t$, the connectivity $d$ of the social network, and the cost of advertising $c$ are the same in both markets. However, the degree of assortativity in these two networked markets is different: in particular, friends’ product preferences exhibit a stronger correlation in market B than in market A. (The consumers of market A and B can also be thought of as members of the WOM networks on two SPs that the firm is considering to run the pre-launch advertising campaign on.) Should the monopolist set the same price and the same advertising level in both markets? How does the assortativity of the network affect the optimal price of the product?

#### 4.4.1 Methods

We note that, due to the implicit dependence of $F(x)$ and $G(z,x)$ on the assortativity in the network through the neighbor distribution $f$, the expression in equation 3 does not lend itself well to standard monotone comparative statics techniques looking at increasing differences. Therefore, to characterize the optimal behavior of the monopolist in terms of the assortativity of the network,
we resort to numerical analysis. Further, for easier exposition, we restrict \( f(x) \) so that assortativity can be captured by a single parameter. In particular, throughout the rest of the paper we assume that

\[
f(x) = \begin{cases} 
-4hx + h + 1 & \text{if } 0 \leq x \leq 0.5 \\
4hx - 3h + 1 & \text{if } 0.5 < x \leq 1,
\end{cases}
\]

where \( h \in [-1,1] \) is the parameter measuring assortativity: \( h > 0 \) corresponds to homophily (Case 1 from section 3), \( h < 0 \) to heterophily (Case 3), and \( h = 0 \) means no assortativity (Case 2). This notation allows us to capture the degree of assortativity in markets A and B in the above example as \(-1 \leq h_A < h_B \leq 1\).

To derive strategic insights that apply across-the-board, we calculate the optima for a wide range of parameters: \( t \in \{1,2,\ldots,10\}, V \in \{1,2,\ldots,9\}, h \in \{-1, -0.99, \ldots -0.01, 0.01, \ldots, 0.99, 1\}, d \in \{1,2,3,\ldots,10,20\}, \) and \( c \in \{1,2,\ldots,10\} \). (Our computational approach is detailed in Appendix B.) Thus, the problem space consists of about one million problem instances, which cover all the managerially important scenarios. Of the above parameters, the regular degree \( d \) may require some explanation. As the average Facebook user has friends well in the triple digits, our values for \( d \) seem fairly small. However, it is important to note that \( d \) corresponds to the number of \textit{actively transmitting} friend relationships of any consumer. Thus, considering that a consumer may not broadcast their product experience to all their friends and/or the possibility of various kinds of noise interfering with the message containing the product information, the range \( 1 \leq d \leq 10 \) is indeed quite realistic.\(^{10}\)

Finally, we also use computers to verify the propositions (3-5) that are developed based on the numerical results. A common property of these propositions is that they take as input a set of problem instances from the problem space described above, and output the distribution of some extreme value (optimal price or advertising, level of other parameter that corresponds to maximum

\(^{10}\)Studying online diffusion on Twitter, Bakshy et al. (2011) and Goel et al. (2012) also find low rates of information transmission.
profits, etc.) as a function of one of the parameters. Thus, after computing the optimal outcome for every point in the problem space, we can easily verify the existence of counterexamples to our propositions computationally.\footnote{In doing so, we have to admit the possibility of “rounding errors,” i.e. that for any extreme value that would actually occur in between grid points, our numerical methods may find the optimum at any corner of the grid cell containing the truly optimal solution of the problem instance.} We note that these arguments are not “computer-aided proofs” in the classical sense of reducing a theorem to a finite number of instances which are then evaluated by a computer (e.g., Robertson et al. (1997)). However, we stress that they are based on precise evaluations of payoffs at a fairly large number of problem instances that cover all managerially relevant cases. We report our results in the next sections.

4.4.2 The Impact of Network Assortativity on Prices

Our numerical analysis shows that the monopolist rarely sets the same price in markets A and B. Figure 4 displays some examples with $t = 10$ and $c = 1$ for various levels of $V$ and $d$. For each subplot, $V$ is kept fixed, and the lines show the benchmark $p = V/2$ and the optimal prices for $d = 2, 5, \text{ and } 20$ in the entire range of $-1 \leq h \leq 1$.

It is clear that the level of discount applied by the monopolist to earn WOM (i) May vary with $h$, (ii) The pattern of this variation depends on $V$ and $d$, and (iii) For certain combinations of $V$ and $d$, the optimal price may vary nonmonotonically with $h$. To be able to explain all the patterns shown in Figure 4, it is important to recall that the reason behind the monopolist’s price adjustments described in Proposition 1 is to trade off some of the margins made in the available market for the extra WOM earned through allowing more consumers in. When the indifferent consumer is expected to have many connections to otherwise unreached consumers in the available market.
then the firm prefers to lower its price to earn more WOM. Based on this, when $z > 0.25$ (i.e., the marginal consumer is on the opposite side of the circle from the firm) then higher assortativity should mean less overall connections from the marginal consumer to the available market, and thus serve as an incentive to raise price. This pattern is indeed confirmed by our analysis.

Similarly, for $z < 0.25$ higher assortativity should increase the incentives to lower price as it increases the connectivity of the marginal consumer to the available market. However, it is possible that those new connections brought in by the consumer at $z$ reach relatively few new customers. In particular, since setting $p = V - t/2$ (or $z = 0.5$) always achieves $Q(\alpha, 0.5) = 1 - (1 - \alpha)^{d+1}$ independently from the degree of assortativity, for any fixed $\alpha$ and $-1 \leq h_A < h_B \leq 1$ there must be some location $z'$ where the marginal profitability of the consumer at $z'$ is less under assortativity $h_B$ than under assortativity $h_A$. When the saturation of advertising (cf. Zubcsek and Sarvary (2011)) in the network is strong (e.g., for high $d$) then $z'$ is still in the most profitable range of $z$, making the monopolist prefer to increase the price for its product as assortativity increases. This is captured by the following proposition.

**Proposition 3.** For a market characterized by $V$, $t$, $c$, $d$, and $h$, if the penetration of the available market ($Q(\alpha, z)/2z$) corresponding to the optimal $\alpha, z$ strategy is low (high) then the monopolist weakly decreases (increases) its price with greater assortativity.  

**Corollary 2.** For any fixed values of $V$, $t$, $c$, and $d$, price may exhibit an increasing, decreasing, or U-shaped (but never an inverse U-shaped) pattern as a function of $h$.

It is interesting to discuss how Proposition 3 explains the pricing patterns shown in Figure 4. For high $V$, we have $z > 0.25$ and as expected, price increases as the network becomes more homophilous. For moderate levels of $V$, $z$ is less than 0.25 yet we see that the monopolist still increases its price with assortativity. Even more curious is the pattern we find for low $V$: for low $d$, the monopolist decreases its price with assortativity but for high degree, the exact opposite may

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12The term “weakly” is only relevant when either $p = V - t/2$ or $p = V/2$. 

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happen. The most interesting pattern happens for $V = 1, d = 20$: When $h$ is low, the monopolist decreases its price with assortativity, while for high $h$, further increasing homophily results in a higher price.

Figure 5 helps explain these seemingly paradoxical phenomena. For $V = 4$, the penetration of the available market, $Q/2z$, is always above 50%, resulting in a strong saturation of WOM, leading to a positive correlation between price and assortativity. For $V = 2$, Figure 5 shows marked differences in $Q/2z$ – this explains why price and assortativity are positively correlated for $d = 2$ and negatively correlated for $d = 5$ and $d = 20$. Finally, for $V = 1$, the available market is very small. By our assumptions on social interactions, this implies that there are less friendships within the available market, reducing the extent to which WOM acts as an “advertising multiplier”. This leads to low levels of advertising, and consequently, penetration of the available market. However, when $d$ is high, even a small increase in the rate of friends within the available market may correspond to huge differences in the probability that a customer is informed. Figure 5 shows that when consumers have 20 friends, then higher levels of homophily do indeed boost the effectiveness of advertising, leading to a rapidly increasing penetration of the available market. This ultimately makes the monopolist raise its price with assortativity.

In sum, the monopolist’s price may decrease or increase with the correlation of friends’ preferences. Both trends happen in response to the increase in the “localness” of WOM externalities caused by higher assortativity. When market penetration is low, higher assortativity increases the expected social value of every consumer in the available market, including that of the marginal consumer. This leads the monopolist to cast a wider net by lowering its price. However, when market penetration is higher, higher assortativity increases the social value of only those customers near the firm. Due to the strong saturation of the advertising information, the social value of consumers near the indifferent customer decreases. In this scenario, the monopolist is better off abandoning the marginal consumer and raising its price. Finally, when $d$ is high and $V$ is low,
these two effects respectively imply that the correlation between price and assortativity is negative at low and positive at high levels of assortativity. This illustrates that in dense social networks it is absolutely essential for firms to incorporate the degree of assortativity in their pricing strategy.

[– Insert Figure 6 around here –]

4.4.3 The Impact of Network Assortativity on Advertising

So far we have focused on optimal prices and only implicitly discussed advertising. However, revisiting our example about markets A and B that only differ in the degree of assortativity, we find that the monopolist typically sets not only different prices but also different advertising levels in the two markets. To illustrate this, we analyzed the examples introduced in Figure 4: the corresponding advertising optima (derived by our numerical analysis) are displayed in Figure 6. The following proposition formalizes our findings.

Proposition 4. For a market characterized by \( V, t, c, d, \) and \( h \), if the penetration of the available market \( Q(\alpha, z)/2z \) corresponding to the optimal \( \alpha, z \) strategy is low (high) then the monopolist weakly increases (decreases) its advertising with greater assortativity.\(^{13}\)

We have covered the basics of the mechanism driving Proposition 4 in the previous section. When \( Q/2z \) is low (also implying that \( \alpha < Q/2z \) is low), then advertising and WOM act as strategic complements: stronger WOM results in higher returns on advertising. As increasing \( h \) indeed increases the reach of WOM in the available market, the monopolist increases \( \alpha \) with \( h \). However, as \( Q/2z \) increases, advertising and WOM become strategic substitutes: advertising becomes less and less effective in the margin due to the increasing redundancy of WOM. Therefore, when market penetration is high, the monopolist responds to a further increase of \( h \) by reducing the costly advertising, leaving more to WOM communication.

\(^{13}\)The term “weakly” is only relevant when either \( p = V - t/2 \) or \( \alpha = 1 \) (cf. Proposition 1).
Corollary 3. For any fixed values of $V, t, c,$ and $d$, advertising may exhibit an increasing, decreasing, or inverse U-shaped (but never a U-shaped) pattern as a function of $h$.

Propositions 3 and 4 follow a similar logic. As assortativity increases, the penetration of the available market corresponding to the monopolist’s optimal strategy also increases, forcing the monopolist to eventually respond by raising its price and decreasing its advertising. However, there is a fundamental difference in how these two actions help increase profitability. Increasing price abandons the consumer on the edge of the available market, the consumer with the smallest social value. Lowering advertising, on the other hand, abandons the average consumer (since advertising cannot be targeted). While at high levels of $Q/2z$, there is a lot of redundancy in WOM, implying that both types of consumer have little incremental social value on their own, this value is nevertheless smaller for the consumer on the edge of the market. Therefore, increasing price becomes an effective tool to increase profitability at a lower level of assortativity than needed to make the lowering of advertising a profitable way to deal with the saturation of WOM.

Corollary 4. The monopolist may not simultaneously decrease both its price and its advertising with assortativity.

4.4.4 The Impact of Overall Connectivity on Advertising

In the previous sections, we concentrated only on the assortativity dimension of network structure, conducting comparative statics analyses keeping the number of social links in the network constant. In this section, we discuss how the overall strength of WOM, captured by our degree ($d$) parameter, impacts advertising strategies.

First, Figure 6 shows that at low baseline valuation $V$, the monopolist advertises more with higher $d$, while at high $V$, the opposite is true. This effect is reminiscent to how the cost of advertising impacts optimal strategies in Galeotti and Goyal (2009). Assuming no difference in $V$, they found that when advertising is efficient (i.e., $c$ is low), then the chosen level of advertising is higher,
leading to more saturation, making advertising and WOM substitutes. On the other hand, when $c$ is high, the monopolist resorts to less advertising. Since this results in less saturation, advertising and WOM are complements in this case. Our results are consistent with these predictions, showing that differences in the baseline valuation lead to different advertising levels and cause similar patterns at the same advertising cost $c$.

More interesting is how the connectivity $d$ affects the shape of the advertising curve. Corollary 3 states that one of the following is true: (i) Advertising is increasing in assortativity, reaching its highest value at $h = h^* = 1$, or (ii) Advertising is increasing for $h < h^*$ and decreasing thereafter, or (iii) Advertising is highest for $h = h^* = -1$, decreasing for all $h \in [-1, 1]$. Thus, advertising has only one local maximum for $h \in [-1, 1]$, allowing us to compare families of social networks of different connectivity in terms of which assortativity level corresponds to the maximum level of advertising (keeping $V$, $t$, and $c$ fixed). In the analysis of optimal advertising levels, we argued that the drop in advertising is caused by the saturation of WOM in the network. It follows that an increase in assortativity affects the monopolist’s strategy more when there are more social relationships in the market. This is formalized in the following proposition:

**Proposition 5.** Higher connectivity amplifies the impact of assortativity on advertising. In particular, the maximum advertising is set at weakly lower levels of $h$ for networks with more social links.

To understand the importance of Proposition 5, let us consider the owner of a social network whose revenues stem from advertising that firms buy to target individual network members. The results of Galeotti and Goyal (2009) show that the network owner may extract more revenues from firms when the (average) valuation of network members is low. The trend of social network portals (e.g., Facebook) focusing on not only growing in terms of members but also developing as a WOM platform (e.g., through “members-you-may-know” services and instant messaging integration) is therefore a profitable move in this regard.
However, Proposition 5 reveals that the degree of assortativity in the network interacts with overall connectivity. Thus, based on the valuation of consumers for the advertised product and the existing degree of homophily in the network, the network owner will benefit differently from encouraging the social interactions of network members with similar vs. different product preferences. In networks with high connectivity and strong homophily, creating further WOM between “birds of different feathers” may indeed produce more advertising revenues to the network than creating WOM “preserving the existing level of homophily”.

5 Competition in Advertising and Prices

In this section, we model two competing risk-neutral firms located on opposite sides of the market. For notational convenience, we assume that Firm 1 is located at 0 while Firm 2 is at 0.5. The two firms simultaneously set their prices and advertising levels. Any consumer may learn about either or both products directly via advertising or indirectly via WOM from any of their neighbors. However, consumer \( i \) only spreads information about product \( j \) (to all their neighbors) if \( i \) in fact prefers to buy product \( j \) (both over buying the competing product and over the outside option). If a consumer learns about both offerings, they buy the product that gives them the highest utility above 0, or stay out of the market if they prefer the outside option to both products.

To draw a comparison to the monopoly case, we uphold the assumption that the entire WOM process unfolds before anyone could make a purchase. We note that while this assumption is not critical in the monopoly case (as long as discounting can be assumed away), herein it is essential that one firm’s WOM be able to beat the competitor’s advertising for our analysis to hold. Thus, the results in this section apply more directly to scenarios wherein the informative advertising released by firms precedes the actual product launch such that the entire WOM process can unfold before any purchases are made.  

\footnote{This is a standard assumption in the analytical literature (Campbell 2013, Domingos and Richardson 2001, Gale-}
Let the advertising and pricing levels chosen by the two firms be denoted by \((\alpha_1, p_1)\) and \((\alpha_2, p_2)\), respectively. Further, let \(z_i = (V - p_i)/t\) indicate the position of the consumers indifferent between Firm \(i\)’s product and the outside option. Without loss of generality, we assume that \(z_1 \geq z_2\), i.e. that the product of Firm 2 is not cheaper than that of Firm 1. We begin the analysis by noting that in markets with low baseline valuation, the firms act as monopolists.

**Observation 1.** If \(z_1 \leq 0.25\) then the two firms price and advertise as monopolists, respectively. This implies that \(\alpha_1 = \alpha_2, z_1 = z_2\), and that Propositions 1-5 hold for the firms’ optimal behavior.

Since we have already characterized the pricing and advertising patterns of the players in such trivial equilibria in section 4, in this section we’ll assume that \(z_1 + z_2 \geq .5\), i.e. that the entire market is covered. Such equilibria correspond to products that enjoy a generally high valuation across the entire market that the firms advertise to. For instance, in a market that has two segments, one of very low and one of sufficiently high valuation, if the firms have the technology to target only the high valuation consumers, then the social network defined as the high-valuation consumers and the friendships between them becomes the WOM network in our model.

The consumer indifferent between the two products is located at \(z^* = (z_1 - z_2)/2 + 0.25\). When the entire market is covered, the three types of indifferent consumers divide the market into four segments, as illustrated in Figure 7. (In the profit formulas, there may be up to six terms as segments 2 and 3 are each comprised of two disconnected sub-segments.) Table 2 summarizes the purchase behavior of these four segments depending on their exposure to the advertising of the competing firms.

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Thus, the demand for Firm 1 can be calculated as the sum of the following four terms:

1. Consumers in segments 1 and 2 (above $z^*$ in Figure 7) who receive Firm 1’s advertising.

2. Consumers in segment 3 (below $z^*$ and above $z_1$) who receive Firm 1’s advertising but not Firm 2’s advertising and who do not have a neighbor who spreads information about Firm 2’s product (which would then be the preferred one). It is implicit in this definition that neighbors of the included consumers in segments 3 and 4 do not receive Firm 2’s message, while their neighbors in segment 2 either do not receive Firm 2’s message or they receive Firm 1’s message as well.

3. Consumers in segments 1 and 2 who do not receive Firm 1’s advertising but have at least one neighbor in segments 1-3 who spreads WOM about Firm 1’s product. This implies that this neighbor does receive Firm 1’s message, and if the only such neighbor is in segment 3 then at least one of these neighbors does not receive Firm 2’s advertising message.

4. Consumers in segment 3 who do not receive advertising at all and receive WOM about Firm 1’s product but not about Firm 2’s product. This component of the demand corresponds to the probability that a consumer does not have a neighbor who spreads Firm 2’s product info while it is not the case that none of their neighbors are spreading any WOM.

Let these four demand components be denoted by $B_1, B_2, B_3,$ and $B_4,$ respectively (we omit the parameters $(\alpha_1, \alpha_2, z_1, z_2)$ to abbreviate notation). Using the formulas for $B_1-B_4$ derived in Appendix C, we can express

$$Q_1(\alpha_1, \alpha_2, z_1, z_2) = B_1 + B_2 + B_3 + B_4; \text{ and}$$
$$\Pi_1(\alpha_1, \alpha_2, z_1, z_2) = (V - t \cdot z_1) \cdot Q_1(\alpha_1, \alpha_2, z_1, z_2) - c \cdot \alpha_1^2. \quad (7)$$

Similarly, $Q_2(\cdot)$ and $\Pi_2(\cdot)$ can be derived by switching every single subscript indexing the two firms in the formulas (including in $z^*$) from 1 to 2 and from 2 to 1.

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Just as in section 4.4, we assume that \( d(x) \equiv d \in \mathbb{N}^+ \) and \( f(x) \) is defined as in equation 6, and proceed with a numerical analysis that is detailed in Appendix B. As the results indicate that the equilibria are, without exception, symmetric (up to a rounding error of a single cell), in the remainder of this section we assume that \( \alpha_1 = \alpha_2 \) and \( z_1 = z_2 \) (implying \( z^* = 0.25 \)). Below we compare the pricing and advertising levels between the monopolist’s optimum and the Pareto optimal symmetric equilibrium identified by our search for the range of \( V, t, c, d, \) and \( h \) for which any equilibrium with \( z_1 + z_2 \geq 0.5 \) is found.

Unlike in the monopoly case wherein the expected demand, and hence the marginal value of consumers at location \( z \) is a differentiable measure, under competition, \( z^* \) (see Figure 7) divides the market into two strikingly different regions that we will denote as the domains of the two firms. In particular, when the market is heavily saturated with product information, neither firm is able to capture many consumers from the other firm’s domain no matter how low they set their price. Intuitively, for the \( z_1 = z_2 \geq 0.25 \) scenario this suggests that the competing firms will often set higher prices than their monopolist counterpart would. This is confirmed by our analysis for most settings: in many cases, the competing firms even set \( p_i > V/2 \). However, we also find that competing firms sometimes set lower prices than the monopolist would in the same market. This happens exclusively when (i) Advertising is expensive (\( c \) is high), (ii) WOM is weak (\( d \) is low), and (iii) The network exhibits strong heterophily (\( h \ll 0 \)).

The intuition behind the above phenomenon is that these three factors combine for a market wherein neither firm is able to profitably reach a high penetration of their own domain. High \( c \) makes the firms settle for less advertising; which is then not only spread to only a few others, but most of the WOM initiated in the firm’s domain is received in the competitor’s domain. On the other hand, the firms’ low market penetration in their own domains means that either firm is able to capture more consumers from their competitor’s domain as long as those prospects have a high enough willingness-to-pay. Moreover, thanks to the heterophilous network structure, consumers in
the competitor’s domain may also help the firm capture more prospects from their own domain via WOM. Thus, the firms have stronger incentives to lower their price below \( p_i = V - t/4 \) and when all three of the above factors are prevalent, the competing firms may even price below the optimal monopoly price.

The natural division of the market into two equal domains also suggests lower returns on advertising, and hence lower advertising levels under competition. This intuitive pattern is indeed confirmed for most settings. However, we also found that when (i) Advertising is cheap \((c)\) is low), (ii) WOM is strong \((d)\) is high), and (iii) The network exhibits strong heterophily \((h \ll 0)\), then the level of advertising set by the competing firms may exceed the optimal monopoly advertising, especially when \( V \) is relatively low (i.e., closer to \( t/2 \)).

The rationale for this behavior is that the first two conditions pose circumstances in which the firms are better off focusing on their own domains by charging higher prices. Moreover, at such higher prices they have stronger incentives to invest into extra advertising, especially when advertising is cheap. However, as discussed in section 4.4.3, at high rates of market penetration, advertising and WOM become substitutes. Therefore, greater assortativity in the network reduces the advertising investment and so it rarely exceeds the monopoly optimum. An exception is when \( V \) is around \( t/2 \) (e.g., \( V = 6, t = 10 \)): as the lower valuation requires a higher penetration of the competing firm’s domain to justify lowering the price to admit consumers from therein, the prices in equilibrium are typically set around \( V - t/4 \), eliminating segments 2 and 3 from Figure 7. In such situations, even under homophily \((h > 0)\) or weak WOM \((low d)\), the firms find it beneficial to buy more advertising than the monopolist, charging much lower prices in optimum, would.

Finally, to see if prices and advertising respond the same way to changes in assortativity both in the monopoly and duopoly settings, we revisited Corollaries 2 and 3 using the results from our numerical analysis. We found that Corollary 2 holds also for a duopoly throughout the entire

\footnote{Recall that in this section we are assuming that the monopoly price is at most \( V - t/4 \).}
problem space specified in section 4.4.1, and that Corollary 3 holds for most settings. However, we also found that for high baseline valuation and weak WOM, advertising may exhibit a U-shaped pattern under competition – something that does not occur in the monopoly setting. Figure 8 illustrates such a case for $V = 9$, $t = 10$, $c = 1$, and $d = 2$: as $h$ increases, advertising first decreases and then increases. The pricing curve displayed in the right half of Figure 8 explains why this happens: $V$ is so high that at $h = -1$, the firms charge the lowest feasible price $p_i = V - t/2$. Since this price is incidentally still very high relative to the cost of advertising, the firms set high advertising levels that lead to a high overall rate of market penetration. As assortativity increases, it increases the strength of within-domain WOM, allowing the firms to reduce their advertising expenditures. At the same time, the increase of $h$ makes it gradually harder for the firms to capture the consumers nearest to their competitor. Eventually the firms give up on these consumers and raise their price. As a consequence, however, they get more bang for their advertising buck, and this ultimately makes them raise advertising as $h$ increases further.

It is interesting to note that while under monopoly, it is also possible to have $p = V - t/2 \leftrightarrow z = 0.5$ admitting every consumer in the market, yet such low prices still may not result in a U-shaped advertising pattern. This is because while $z$ is kept constant at 0.5, our model is equivalent to that of Galeotti and Goyal (2009), and so the advertising optimum is constant in assortativity (cf. section 4.1.2). Thus, in the monopoly setting, we may observe a pattern wherein advertising is first constant and then increases, but not a pattern wherein advertising first decreases and then increases.
6 Conclusion

Social advertising is one of the fastest-emerging formats to promote new products. This is in no small part due to the unique feature that SPs are both the medium for advertising to consumers and the medium hosting the subsequent word-of-mouth interactions between them. It is commonly accepted in the literature that information on the network structure of WOM – something readily available at companies like Facebook – can benefit both advertisers and the owner of the WOM platform. However, while it is also well-documented that the preferences of neighbors in social networks tend to be correlated, neither industry best practices nor the extant literature speak to incorporating network assortativity into the pricing and advertising strategies of marketers. Instead, both the industry and the literature on advertising to social networks has used the working assumption that consumers’ WOM behavior is independent from their reservation price for the advertised product. This assumption has two important consequences. First, the optimal strategies of marketers are independent from the correlation of product preferences between friends in the WOM network. Second, the pricing decision of the firm is independent from the subsequent advertising decision, and the optimal price is simply the price that the firm would set if there was no WOM at all in the market.

Intuitively, consumers’ WOM behavior should depend on their reservation price – for example, it is realistic to assume that consumers who do not try the advertised product will neither spread WOM about it, rendering false a key assumption in most studies on advertising to social networks. As information on the correlation of network neighbors’ preferences can often be obtained efficiently, the lack of understanding how firms should act on the assortativity of the WOM network has been an important gap in the literature.

Our paper addresses this gap by constructing a model of informative advertising to a social network that may exhibit homophily or heterophily. We assume that firms may only rely on WOM
spread by those consumers who are both reached by advertising and have a reservation price above the advertised price of the product. For these settings, we derive the optimal strategy both for a monopolist marketer and for two, horizontally differentiated competing firms. We find that under most circumstances, blanket advertising is wasteful and that in response to this, the monopolist should not only not advertise to every consumer but also set a price lower than the naïve optimum derived in extant analytical work. The only exception to this applies to highly differentiated brands that have been able to transform the structure of assortativity in the network such that high- and low-valuation consumers form two, essentially disjoint groups. We show that such highly differentiated brands actually may prefer to set a price that exceeds the naïve optimum benchmark.

For the case of undifferentiated brands, we also find that both the optimal price and advertising may exhibit a non-monotonic pattern in response to the degree of assortativity in the market. In particular, we find that holding all other parameters constant, the increase of assortativity may trigger a U-shaped pattern of monopoly price or an inverse U-shaped pattern of monopoly advertising. Further, we show that as the overall connectivity of the network (the strength of WOM) increases, the maximal advertising choice of the monopolist occurs in networks with lower assortativity. Finally, for competing firms, we find that they should typically set higher prices and lower advertising than the monopolist. However, we also identify market conditions in which the competing firms prefer to either charge lower prices or set higher advertising than the monopolist would in the same market.

Our findings have several important implications. First and foremost, we show how accurate measures of preference homophily in networked markets can help firms calibrate both their price and advertising level to maximize the return on advertising campaigns. These findings are especially relevant for players in the social advertising ecosystem where the infrastructure of the network platform allows firms to obtain information on the assortativity of the network at a low cost. Second, our main pricing result is applicable far beyond the realm of social or even on-
line advertising. It applies to any monopolist marketer that has information about the distribution of willingness-to-pay in the targeted market (say, via surveying a representative sample of consumers). Assuming that the firm cannot afford a campaign that directly informs every consumer in this market, our result implies that the firm should set a lower price than the one that maximizes the area below the demand curve. In doing so, it can accommodate more WOM, leading to a gain in demand that outweighs the losses caused by the lower margins per customer. Our paper is the first in the literature to show this to hold for a broad class of functions characterizing network assortativity. We also show that models employed in prior literature are more adequate to address the implications of assortativity in markets whose connectivity structure is already influenced by the brand of the monopolist (e.g., via consumers’ use of the brand in social interactions). Third, we derive insights for SP owners, identifying circumstances in which the most profitable strategy to grow the network may be to create relationships between “birds of different feathers.” This result is particularly important since many friendship recommendation engines simply search among the second-degree contacts (or “friends of friends”) – boosting the degree of assortativity of already homophilous networks further in the process.

We conclude by acknowledging some limitations of our work. First, by extending the model of Galeotti and Goyal (2009), we assumed away the possibility that two neighboring nodes would have any neighbors in common, focusing only on networks with zero clustering. As long as both advertising and WOM are purely informative, this is a trivial restriction. However, it is important to bear in mind that even if advertising is purely informative, as soon as WOM may be, at least to some extent, persuasive (say, some high-WTP consumers decide to buy the product only when they hear positive reviews of the product from at least two of their neighbors) then this assumption does reduce the applicability of our findings. Second, throughout our analysis, we focused exclusively on regular networks. This limitation is also minor, since solving the firm’s problem on more complex networks may simply be viewed as a compound problem instance arising from the
convex combination of some regular networks. On the other hand, keeping the networks regular simplified the analysis and the presentation of our results. Finally, we assumed the assortativity of the WOM network to be exogenous. In practice, however, some firms could manipulate the design (in particular, the relevant dimensions of product characteristics) of their product so that they strategically increase or reduce the assortativity of the network of consumers. However, we note that as long as the product design is finished before the firm sets its price and advertising, our analysis still may provide important input to the pricing and advertising strategy for the product.

To summarize, we have developed a novel approach to study how the positive or negative correlation of friends’ reservation prices for a product should affect the optimal price and advertising levels for that product. Our work has important general implications and further insights specific to the social advertising industry. Despite the limitations acknowledged above, we hope that our paper will spur further research on this increasingly important topic.
References


Figures and Tables

Figure 1: Illustrating how price impacts WOM. The black nodes belong to the available market $I$, the white nodes to the consumers $O$ who are priced out of the market by the firm. Solid lines between consumers indicate WOM links that run within $I$, while dashed lines display relationships that may not benefit the firm. The indifferent consumers are located at $z > z'$ (thus, the higher price is set in the market on the right).

Figure 2: The probability that a given friend of consumer $x$ has both received advertising and has a low enough reservation price to spread WOM is $\alpha \cdot [F(z + x) + F(z - x)]$. 
Figure 3: Illustrating a peculiarity of the two-type model of assortativity: the friendship networks of consumers B and C are very different, despite the similarity of their brand valuations.
Figure 4: Optimal monopoly price at various levels of baseline valuation, degree and assortativity (at $t = 10, c = 1$). The line at the top of each chart indicates the optimal naïve price derived from the prior models discussed in section 4.1.
Figure 5: Penetration of the available market corresponding to the optimal strategy of the monopolist at various levels of baseline valuation, degree and assortativity (at $t = 10, c = 1$).
Figure 6: Optimal advertising of the monopolist at various levels of baseline valuation, degree and assortative (at $t = 10, c = 1$).
Figure 7: Illustrating the market segments determined by the three types of indifferent consumers.

Figure 8: Optimal advertising and pricing strategies in the symmetric competitive equilibria for $V = 9$, $t = 10$, $c = 1$, $d = 2$
Table 1: Illustrating the differences between the assortativity models of the present work versus Campbell (2013) using Figure 3

<table>
<thead>
<tr>
<th>Pair of Nodes</th>
<th>Similarity in the present work</th>
<th>Similarity in Campbell (2013)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A,B</td>
<td>Medium</td>
<td>High</td>
</tr>
<tr>
<td>A,C</td>
<td>Medium</td>
<td>Low</td>
</tr>
<tr>
<td>A,D</td>
<td>Low</td>
<td>Low</td>
</tr>
<tr>
<td>B,C</td>
<td>High</td>
<td>Low</td>
</tr>
<tr>
<td>B,D</td>
<td>Medium</td>
<td>Low</td>
</tr>
<tr>
<td>C,D</td>
<td>Medium</td>
<td>High</td>
</tr>
</tbody>
</table>

Table 2: Consumer choice in the four segments from Figure 7 as a function of advertising reach

<table>
<thead>
<tr>
<th>Advertising reach</th>
<th>Segment 1</th>
<th>Segment 2</th>
<th>Segment 3</th>
<th>Segment 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>No advertising reaches consumer</td>
<td>Ø</td>
<td>Ø</td>
<td>Ø</td>
<td>Ø</td>
</tr>
<tr>
<td>Only Firm 1 reaches consumer</td>
<td>Product 1</td>
<td>Product 1</td>
<td>Product 1</td>
<td>Ø</td>
</tr>
<tr>
<td>Only Firm 2 reaches consumer</td>
<td>Ø</td>
<td>Product 2</td>
<td>Product 2</td>
<td>Product 2</td>
</tr>
<tr>
<td>Both firms reach consumer</td>
<td>Product 1</td>
<td>Product 1</td>
<td>Product 2</td>
<td>Product 2</td>
</tr>
</tbody>
</table>
Appendix A - Proofs

Proof of Lemma 1: To show that lowering price increases the (percentage) penetration of the available market, we will show that for any neighbor distribution \( F : [0, 1] \to [0, 1] \) that respects one of the cases listed in section 3, \( d \in \mathbb{N}^+ \), \( 0 < z < 0.5 \) and \( 0 < \alpha < 1 \),

\[
\frac{\partial Q(\alpha, z)}{\partial z} > 0. \tag{9}
\]

From Equation 1 we get that

\[
\begin{align*}
Q(\alpha, z)/2z &= 1 - (1 - \alpha) \cdot \int_{0}^{z} [1 - \alpha \cdot G(z, x)]^d dx \\
&= 1 - \frac{1 - \alpha}{z} \cdot \int_{0}^{z} [1 - \alpha \cdot (F(z + x) + F(z - x))]^d dx.
\end{align*}
\]

We prove the claim through the three cases, distinguished by the degree of assortativity in the network.

Case 1: Homophily \( (f'(x) < 0 \text{ for } 0 < x < 0.5) \).

In the case of homophily, \( F(x) \) is concave on \((0, 0.5)\). Using that also \( f(x) = f(1-x) \), it is straightforward to derive that for any \( 0 < x \leq z < 0.5 \),

\[
F(z) + F(z) > F(z + x) + F(z - x). \tag{10}
\]

Thus,

\[
\frac{Q(\alpha, z)}{2z} = 1 - \frac{1 - \alpha}{z} \cdot \int_{0}^{z} [1 - \alpha \cdot (F(z + x) + F(z - x))]^d dx < 1 - (1 - \alpha) \cdot [1 - 2\alpha \cdot F(z)]^d dx.
\]

Let

\[
\phi = 1 - (1 - \alpha) \cdot [1 - 2\alpha \cdot F(z)]^d dx - \frac{Q(\alpha, z)}{2z} > 0.
\]
Next, we note that for any $0 < \varepsilon < 0.5 - z$,

$$Q(\alpha, z + \varepsilon) = 2 \cdot (z + \varepsilon) - 2 \cdot (1 - \alpha) \cdot \int_{0}^{z+\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx$$

$$= 2 \cdot (z + \varepsilon) - 2 \cdot (1 - \alpha) \cdot \left\{ \int_{0}^{\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx + \int_{\varepsilon}^{z+\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx \right\}.$$ 

Further, for any $0 < x < 0.5 - z$, $F(z + \varepsilon + x) + F(z + \varepsilon - x) > F(z + 2\varepsilon) + F(z)$, and so

$$\int_{0}^{\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx < \varepsilon \cdot [1 - \alpha \cdot (F(z + 2\varepsilon) + F(z))]^d \Rightarrow \int_{0}^{\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx < \varepsilon \cdot [1 - 2\alpha \cdot F(z)]^d$$

and since

$$\varepsilon - (1 - \alpha) \cdot \varepsilon \cdot [1 - 2\alpha \cdot F(z)]^d \, dx = \varepsilon \cdot \left( \phi + \frac{Q(\alpha, z)}{2z} \right),$$

it must be that

$$\varepsilon - (1 - \alpha) \cdot \int_{0}^{\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx > \varepsilon \cdot \left( \phi + \frac{Q(\alpha, z)}{2z} \right).$$

Finally, since $F(z + \varepsilon + x) + F(z + \varepsilon - x) > F(z + x) + F(z - x)$ for that $F$ is increasing, we also have

$$\int_{\varepsilon}^{z+\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d \, dx < \int_{0}^{z} [1 - \alpha \cdot (F(z + x) + F(z - x))]^d \, dx.$$ 

Thus,

$$Q(\alpha, z + \varepsilon) > 2\varepsilon \cdot \left( \phi + \frac{Q(\alpha, z)}{2z} \right) + Q(\alpha, z).$$

Therefore,

$$\frac{\partial Q(\alpha, z)}{\partial z} \bigg|_{2z} = \lim_{\varepsilon \to 0} \frac{Q(\alpha, z + \varepsilon) - Q(\alpha, z)}{2(\varepsilon + \varepsilon)} \bigg|_{2z} = \lim_{\varepsilon \to 0} \frac{2\varepsilon \left( \phi + \frac{Q(\alpha, z)}{2z} \right) + 2z \cdot \frac{Q(\alpha, z)}{2z} - Q(\alpha, z)}{2z} = \lim_{\varepsilon \to 0} \frac{2\varepsilon \phi}{2(\varepsilon + \varepsilon)} + \frac{\left( \frac{Q(\alpha, z)}{2z} - \frac{Q(\alpha, z)}{2z} \right)}{\varepsilon} = \frac{\phi}{z} > 0.$$
Case 2: No assortativity.

When \( h = 0 \), then for any \( 0 \leq x \leq z < 0.5 \) we have \( F(z+x) + F(z-x) = F(z) + F(z) = 2z \), so

\[
Q(\alpha, z)/2z = 1 - (1 - \alpha) \cdot [1 - \alpha \cdot 2z]^d.
\]

Thus,

\[
\frac{\partial Q(\alpha, z)}{\partial z} = \lim_{\varepsilon \to 0} \frac{Q(\alpha, z + \varepsilon) - Q(\alpha, z)}{2(z + \varepsilon) - 2z} = \lim_{\varepsilon \to 0} \frac{(1 - \alpha) \cdot \left\{ [1 - \alpha \cdot 2z]^d - [1 - \alpha \cdot 2(z + \varepsilon)]^d \right\}}{\varepsilon}
\]

\[
= -(1 - \alpha) \cdot \frac{\partial (1 - \alpha \cdot 2z)^d}{\partial z}
\]

\[
= (1 - \alpha) \cdot d \cdot (1 - \alpha \cdot 2z)^{d-1} \cdot 2\alpha
\]

\[
> 0.
\]

Case 3: Heterophily \((f'(x) > 0 \text{ for } 0 < x < 0.5)\).

The proof is similar to the that of Case 1. However, in the case of heterophily, \( F(x) \) is convex on \((0, 0.5)\). Adding that also \( f(x) = f(1-x) \), one arrives to that for any \( 0 \leq x < z < 0.5 \),

\[
F(2z) + F(0) > F(z + x) + F(z - x). \quad (11)
\]

Thus,

\[
\frac{Q(\alpha, z)}{2z} = 1 - \frac{1 - \alpha}{z} \cdot \int_0^z [1 - \alpha \cdot (F(z + x) + F(z - x))]^d dx < 1 - (1 - \alpha) \cdot [1 - \alpha \cdot F(2z)]^d dx.
\]

Let

\[
\phi = 1 - (1 - \alpha) \cdot [1 - \alpha \cdot F(2z)]^d dx - \frac{Q(\alpha, z)}{2z} > 0.
\]

Next, we note that for any \( 0 < \varepsilon < 0.5 - z \),

\[
Q(\alpha, z + \varepsilon) = 2 \cdot (z + \varepsilon) - 2 \cdot (1 - \alpha) \cdot \int_0^{z+\varepsilon} [1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x))]^d dx = 2 \cdot (z + \varepsilon) - 2 \cdot (1 - \alpha) \cdot \left\{ \int_0^z [1 - \alpha \cdot (F(z + x) + F(z - x))]^d dx + \right.
\]

\[
\left. \int_z^{z+\varepsilon} [1 - \alpha \cdot (F(z + x) + F(z - x))]^d dx \right\}.
\]

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Further, for any $z < x < z + \varepsilon < 0.5$, $F(z + \varepsilon + x) + F(z + \varepsilon - x) > F(2z + \varepsilon) + F(0) > F(2z)$, and so
\[
\int_{z}^{z+\varepsilon} \left[ 1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x)) \right]^d \, dx < \varepsilon \cdot [1 - \alpha \cdot F(2z)]^d
\]
and since
\[
\varepsilon - (1 - \alpha) \cdot \varepsilon \cdot [1 - \alpha \cdot F(2z)]^d \, dx = \varepsilon \cdot \left( \phi + \frac{Q(\alpha, z)}{2z} \right),
\]
it must be that
\[
\varepsilon - (1 - \alpha) \cdot \int_{z}^{z+\varepsilon} \left[ 1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x)) \right]^d \, dx > \varepsilon \cdot \left( \phi + \frac{Q(\alpha, z)}{2z} \right).
\]
Finally, since $F(z + \varepsilon + x) + F(z + \varepsilon - x) > F(z + x) + F(z - x)$ for that $F$ is increasing, we also have
\[
\int_{0}^{z} \left[ 1 - \alpha \cdot (F(z + \varepsilon + x) + F(z + \varepsilon - x)) \right]^d \, dx < \int_{0}^{z} \left[ 1 - \alpha \cdot (F(z + x) + F(z - x)) \right]^d \, dx.
\]
Thus,
\[
Q(\alpha, z + \varepsilon) > 2\varepsilon \cdot \left( \phi + \frac{Q(\alpha, z)}{2z} \right) + Q(\alpha, z).
\]

Therefore,
\[
\frac{\partial Q(\alpha, z)}{\partial z} \bigg|_{z} = \lim_{\varepsilon \to 0} \frac{Q(\alpha, z + \varepsilon) - Q(\alpha, z)}{2(z+\varepsilon)} = \lim_{\varepsilon \to 0} \frac{2\varepsilon \left( \phi + \frac{Q(\alpha, z)}{2z} \right) + 2z \cdot \frac{Q(\alpha, z)}{2z}}{2(z+\varepsilon)} - \frac{Q(\alpha, z)}{2z} = \lim_{\varepsilon \to 0} \frac{2\varepsilon \phi}{2(z+\varepsilon)} - \frac{Q(\alpha, z)}{2z} < \frac{2\varepsilon \phi}{2(z+\varepsilon)} + \frac{Q(\alpha, z)}{2z} = \frac{\phi}{z} > 0.
\]
This concludes the proof. 

**Corollary 5.** For any $h \in [-1, 1]$, $d \in \mathbb{N}^+$, $0 < z < 0.5$ and $0 < \alpha < 1$,
\[
0 > z \cdot \left\{ [1 - \alpha \cdot G(z, \varepsilon)]^d + d \cdot \int_{0}^{z} [1 - \alpha \cdot G(z, \varepsilon)]^d \, dx \right\} - \int_{0}^{z} [1 - \alpha \cdot G(z, \varepsilon)]^d \, dx.
\]
Proof of Corollary 5: Recall that

\[ Q(\alpha, z)/2z = 1 - (1 - \alpha) \cdot \int_{0}^{z} \frac{[1 - \alpha \cdot G(z, x)]^{d} dx}{z}. \]

Thus, for \( 1 - \alpha > 0 \), evaluating the derivative and multiplying both sides by \( z^2 > 0 \), we get that

\[ \frac{\partial Q(\alpha, z)}{\partial z} > 0 \Leftrightarrow 0 > \frac{\partial \int_{0}^{z} [1 - \alpha \cdot G(z, x)]^{d} dx}{z} \Leftrightarrow 0 > z \cdot \left\{ [1 - \alpha \cdot G(z, z)]^{d} + d \cdot \int_{0}^{z} [1 - \alpha \cdot G(z, x)]^{d-1} \cdot \left( -\alpha \cdot \frac{\partial G(z, x)}{\partial z} \right) dx \right\} - \int_{0}^{z} [1 - \alpha \cdot G(z, x)]^{d} dx. \]

\[ \Box \]

Proof of Proposition 1: The lower bound in the claim is obvious since for \( V - t/2 > 0 \) setting \( p = V - t/2 \) accommodates every consumer in the market. Therefore, herein we focus on the comparison between \( p \) and \( V/2 \).

To prove the first half of the claim, we can show that for any price \( p' > V/2 \) and \( z' = (V - p')/t \), \( \Pi(\alpha, V/2t) > \Pi(\alpha, z') \). To see this, denote the profit to advertising level \( \alpha \) and indifferent consumer \( z \) under no WOM (cf. section 4.1.1) by \( \Psi(\alpha, z) \).

Calculating \( Q(\alpha, z) \) as in Equation 1, we get that for any \((\alpha, z)\),

\[ \Pi(\alpha, z) = \Psi\left( \frac{Q(\alpha, z)}{2z}, z \right). \]

Just as in section 4.1.1, it is trivial to establish that

\[ \Psi\left( \frac{Q(\alpha, V/2t)}{V/2t}, V/2t \right) > \Psi\left( \frac{Q(\alpha, V/2t)}{V/t}, z' \right). \]

Further, Lemma 1 implies that \( Q(\alpha, V/2t)/(V/t) > Q(\alpha, z')/(2z') \), and so

\[ \Psi\left( \frac{Q(\alpha, V/2t)}{V/t}, z' \right) > \Psi\left( \frac{Q(\alpha, z')}{2z'}, z' \right). \]
Thus,
\[
\Pi(\alpha, V/2t) = \Psi\left(\frac{Q(\alpha, V/2t)}{V/t}, V/2t\right) > \Psi\left(\frac{Q(\alpha, z')}{2z'}, z'\right) = \Pi(\alpha, z'),
\]

implying \( p \leq V/2 \) for the optimal price. This obtains the first half of the proof.

Further, if \( \alpha = 1 \) then every consumer is informed and \( p = V/2 \) is clearly optimal. However, since
\[
\frac{\partial \Pi}{\partial \alpha}\bigg|_{\alpha=1, p=V/2} = -2c\alpha + 2 \cdot (V - tz) \cdot \left\{ \int_0^z [1 - \alpha \cdot G(z, x)]^d dx - (1 - \alpha) \cdot \int_0^z [1 - \alpha \cdot G(z, x)]^{d-1} \cdot (-G(z, x)) dx \right\} = -2c + 2 \cdot (V - tz) \cdot \int_0^z [1 - G(z, x)]^d dx = -2c + V \cdot \int_0^{V/2t} \left[ 1 - G\left(\frac{V}{2t}, x\right)\right]^d dx,
\]
then if the lower bound on \( c \) holds, we have \( \alpha < 1 \).

Finally,
\[
\frac{\partial \Pi}{\partial z}\bigg|_{z=V/2t} = -2t \cdot \left\{ z - (1 - \alpha) \cdot \int_0^z [1 - \alpha \cdot G(z, x)]^d dx \right\} + 2(V - tz) \cdot \left\{ 1 - (1 - \alpha) \cdot \left[ 1 - \alpha \cdot G(z, z) \right]^d \right. + \left. \int_0^z d \cdot [1 - \alpha \cdot G(z, x)]^{d-1} \cdot \left( -\alpha \cdot \frac{\partial G(z, x)}{\partial z} \right) dx \right\} = 2t \cdot (1 - \alpha) \cdot \left\{ \int_0^z [1 - \alpha \cdot G(z, x)]^d dx \right. - z \cdot \left[ 1 - \alpha \cdot G(z, z) \right]^d + \left. \int_0^z d \cdot [1 - \alpha \cdot G(z, x)]^{d-1} \cdot \left( -\alpha \cdot \frac{\partial G(z, x)}{\partial z} \right) dx \right\},
\]
and therefore
\[
\frac{\partial \Pi}{\partial z}\bigg|_{z=V/2t, 0 < \alpha < 1} > 0
\]
follows from Corollary 5. Since \( \alpha > 0 \) is obvious, this concludes the proof. \( \square \)
PROOF OF PROPOSITION 2:

Under the two-type model of assortativity, the profits to the \((\alpha, z)\) strategy of the firm depend on whether \(z > s\). To derive the profits to the monopolist, we therefore consider the following two cases.

Case I: \(z \leq s\). When the firm sets a price so high that it accommodates only “high type” consumers, then for any \(-z \leq x \leq z\),

\[
\Pr[x \text{ learns about the product}] = \alpha + (1 - \alpha) \left[ 1 - \left\{ 1 - \frac{\alpha e z}{s} \right\}^d \right],
\]

and the demand in this case is simply

\[
Q_H(\alpha, z) = \int_{-z}^{z} \Pr[x \text{ learns about the product}] \, dx
= 2z \cdot \Pr[x \text{ learns about the product}]
= 2z - 2z \cdot (1 - \alpha) \cdot \left\{ 1 - \frac{\alpha e z}{s} \right\}^d,
\]

and therefore

\[
\Pi_H(\alpha, z) = p \cdot Q_H(\alpha, z) - c \alpha^2
= 2(V - tz) \cdot \left\{ z - z \cdot (1 - \alpha) \cdot \left\{ 1 - \frac{\alpha e z}{s} \right\}^d \right\} - c \alpha^2.
\]

Case II: \(z > s\). When the price is low enough to admit some “low type” consumers, the situation is somewhat more complicated. First, for a high type consumer \(x_1 \leq s\), the probability that one of their randomly chosen friends \(y_1\) gets the firm’s advertising message is

\[
\Pr[y_1 \text{ gets the firm’s advertising}] = \alpha \cdot \left[ e + (1 - e) \cdot \frac{z - s}{0.5 - s} \right].
\]

For a low type consumer \(s < x_2 \leq z\), the probability that one of their randomly chosen friends \(y_2\) gets the firm’s advertising message is

\[
\Pr[y_2 \text{ gets the firm’s advertising}] = \alpha \cdot \left[ \frac{2s \cdot (1 - e)}{1 - 2s} + \frac{1 - 2s \cdot (2 - e)}{1 - 2s} \cdot \frac{z - s}{0.5 - s} \right].
\]
Therefore, the demand for the firm’s product is

\[
Q_L(\alpha, z) = 2 \cdot \int_0^s \left( \alpha + (1 - \alpha) \cdot \left( 1 - \left\{ \frac{e + (1 - e) \cdot \frac{z - s}{0.5 - s}}{1 - 2s} \right\} \right) \right) dx + \\
2 \cdot \int_s^z \left( \alpha + (1 - \alpha) \cdot \left( 1 - \left\{ \frac{2s \cdot (1 - e) + 1 - 2s \cdot (2 - e) \cdot \frac{z - s}{0.5 - s}}{1 - 2s} \right\} \right) \right) dx
\]

\[
= 2z - 2 \cdot (1 - \alpha) \cdot \left( \frac{2s \cdot (1 - e)}{1 - 2s} + \frac{1 - 2s \cdot (2 - e) \cdot \frac{z - s}{0.5 - s}}{1 - 2s} \right) d\\n(z - s) \cdot \left( 1 - \alpha \cdot \left\{ \frac{2s \cdot (1 - e)}{1 - 2s} + \frac{1 - 2s \cdot (2 - e) \cdot \frac{z - s}{0.5 - s}}{1 - 2s} \right\} \right)
\]

and therefore

\[
\Pi_L(\alpha, z) = p \cdot Q_L(\alpha, z) - c\alpha^2
\]

\[
= 2(V - tz) \cdot z - (1 - \alpha) \cdot \left( \frac{2s \cdot (1 - e)}{1 - 2s} + \frac{1 - 2s \cdot (2 - e) \cdot \frac{z - s}{0.5 - s}}{1 - 2s} \right) d\\n(z - s) \cdot \left( 1 - \alpha \cdot \left\{ \frac{2s \cdot (1 - e)}{1 - 2s} + \frac{1 - 2s \cdot (2 - e) \cdot \frac{z - s}{0.5 - s}}{1 - 2s} \right\} \right) - c\alpha^2.
\]

Finally, let

\[
i_{sz} = \begin{cases} 
1 & \text{if } z \leq s, \text{ and} \\
0 & \text{if } z > s.
\end{cases}
\]

Then the monopolist’s optimization problem is setting

\[
(\alpha^*, z^*) = \arg \max_{\alpha \in [0,1], z \in [0,0.5]} i_{sz} \cdot \Pi_H(\alpha, z) + (1 - i_{sz}) \cdot \Pi_L(\alpha, z).
\]

Now set \( V > 0, t > V, 0 < d \in \mathbb{Z}, \) and \( 0 < c. \) We will now construct a structure of assortativity \((e, s)\) at which the monopolist prefers to charge a price \( p > V/2. \) Let \( e = 1. \) Then

\[
\Pi_H(\alpha, z) = 2(V - tz) \cdot z - \frac{z \cdot (1 - \alpha) \cdot \left\{ \frac{\alpha z}{s} \right\}}{d} - c\alpha^2,
\]

and

\[
\Pi_L(\alpha, z) = -c\alpha^2 + 2(V - tz) \cdot z - (1 - \alpha) \cdot \left\{ s \cdot \left\{ 1 - \alpha \right\}^d + (z - s) \cdot \left\{ 1 - \alpha \cdot \left\{ \frac{z - s}{0.5 - s} \right\} \right\}^d \right\}.
\]
We will construct $s$ using the series $(\varepsilon_i), (a_i), (s_i), (\alpha_i), (z_i), (\phi_i), i \in \mathbb{N}$ described below, and show that the monopolist prefers to set $p = V - ts$ over any price larger than $V/2$.

Let $\varepsilon_0 = a_0 = V, s_0 = V/2t$, and for $s = s_0$, let

$$(\alpha_0, z_0) = \arg\max_{z \in [V/2t, 0.5], \alpha \in [0, 1]} \Pi_L(\alpha, z).$$

Let $\phi_0 = 2t \cdot z_0 - V$, and recursively define

\[
a_i = \frac{\alpha_{i-1} \cdot \phi_{i-1} + V \cdot (1 - \alpha_{i-1}) \cdot \left[\left(1 - \alpha_{i-1} \cdot \frac{\phi_{i-1} + V}{t}\right)^d - (1 - \alpha_{i-1})^d\right]}{2 \cdot \left\{1 - (1 - \alpha_{i-1})^{d+1}\right\}},
\]

\[
\varepsilon_i = \min(a_i, \frac{\varepsilon_{i-1}}{2}),
\]

\[
s_i = \frac{V - \varepsilon_i}{2t},
\]

\[
(\alpha_i, z_i) = \arg\max_{s = s_i, z \in [V/2t, 0.5], \alpha \in [0, 1]} \Pi_L(\alpha, z),
\]

\[
\phi_i = 2t \cdot z_i - V \Leftrightarrow z_i = \frac{V + \phi_i}{2t}
\]

**Lemma 2.** If $\exists i > 0$ such that

\[
\varepsilon_i < \frac{\alpha_i \cdot \phi_i + V \cdot (1 - \alpha_i) \cdot \left[\left(1 - \alpha_i \cdot \frac{\phi_i + V}{t}\right)^d - (1 - \alpha_i)^d\right]}{1 - (1 - \alpha_i)^{d+1}},
\]

then we have

$$\Pi_H(\alpha_i, s_i) > \Pi_L(\alpha_i, z_i),$$

implying that the monopolist prefers the price $p = V - ts_i > V/2$ over any price that is at most $V/2$.

**Proof of Lemma 2:** Since $\forall i > 0$, both $\phi_i + V < t$ and $V > \varepsilon_i$,

$$\frac{\phi_i + V}{t} > \frac{\phi_i + \varepsilon_i}{t - V + \varepsilon_i},$$
and so the condition in Inequality 12 implies that

\[
\varepsilon_i \cdot \left[ 1 - (1 - \alpha_i)^{d+1} \right] \quad < \quad \alpha_j \cdot \phi_i + V \cdot (1 - \alpha_i) \cdot \left( 1 - \alpha_i \cdot \frac{\phi_i + \varepsilon_i}{t - V + \varepsilon_i} \right)^d - (1 - \alpha_i)^d
\]

\[
\Rightarrow \quad \varepsilon_i \cdot \left[ 1 - (1 - \alpha_i)^{d+1} \right] \quad < \quad \alpha_j \cdot \phi_i + V \cdot (1 - \alpha_i) \cdot \left( 1 - \alpha_i \cdot \frac{\phi_i + \varepsilon_i}{t - V + \varepsilon_i} \right)^d - (1 - \alpha_i)^d
\]

\[
\Rightarrow \quad 0 \quad < \quad (V - \phi_i) \cdot (1 - \alpha_i) \cdot \left( 1 - \alpha_i \cdot \frac{\phi_i + \varepsilon_i}{t - V + \varepsilon_i} \right)^d - (V - \varepsilon_i) \cdot (1 - \alpha_i)^{d+1}
\]

Now let \( \Delta_i = \Pi_H(\alpha_i, s_i) - \Pi_L(\alpha_i, z_i) \). Then

\[
\Delta_H = 2(V - ts_i) \cdot s_i \cdot \left[ 1 - (1 - \alpha_i)^{d+1} \right] - 2(V - tz_i) \cdot \left\{ z_i - (1 - \alpha_i) \cdot \left[ s_i \cdot (1 - \alpha_i)^d + (z_i - s_i) \cdot \left( 1 - \alpha_i \cdot \frac{z_i - s_i}{0.5 - s_i} \right)^d \right] \right\}
\]

\[
= 2V \cdot (s_i - z_i) + 2t \cdot (z_i^2 - s_i^2) + s_i \cdot (1 - \alpha_i)^{d+1} \cdot 2t \cdot (s_i - z_i) + 2(V - tz_i) \cdot (1 - \alpha_i) \cdot (z_i - s_i) \cdot \left\{ 1 - \alpha_i \cdot \frac{z_i - s_i}{0.5 - s_i} \right\}^d.
\]

\[
= 2(z_i - s_i) \cdot \left( t \cdot (z_i + s_i) - V + (V - tz_i) \cdot (1 - \alpha_i) \cdot \left\{ 1 - \alpha_i \cdot \frac{z_i - s_i}{0.5 - s_i} \right\}^d - ts_i \cdot (1 - \alpha_i)^{d+1} \right)
\]

\[
= (\phi_i + \varepsilon_i) \cdot \left( \phi_i - \varepsilon_i + (V - \phi_i) \cdot (1 - \alpha_i) \cdot \left\{ 1 - \alpha_i \cdot \frac{\phi_i + \varepsilon_i}{t - V + \varepsilon_i} \right\}^d - (V - \varepsilon_i) \cdot (1 - \alpha_i)^{d+1} \right),
\]

since \( z_i - s_i = (\phi_i + \varepsilon_i)/2t \). Therefore, \( \Delta_i > 0 \), concluding the proof of Lemma 2.

Finally, observe that since \( \forall i, \varepsilon_i < \varepsilon_1/2^{i-1} \), \( \lim_{i \to \infty} \varepsilon_i = 0 \). Therefore, \( \lim_{i \to \infty} s_i = V/2t \). Furthermore, as \( \Pi_L(\alpha, z) \), \( z > V/2t \) is continuously differentiable in the implicit parameter \( s \leq V/2t \), we also get that \( \lim_{i \to \infty} (\alpha_i, z_i) = (\alpha_0, z_0) \), and so \( \lim a_i = a_1 \). Thus, there must be some \( i^* > 1 \) for which \( |a_i - a_1| < \varepsilon_1/2 \), implying \( a_r > \varepsilon_1/2 > \varepsilon_{r+1} \). Thus, for \( \varepsilon_{r+1} \), Lemma 2 must hold. This concludes the proof of Proposition 2.

\[ \Box \]
Appendix B - Computational Methods

Monopoly

From equation 6, it is straightforward to calculate $F(x)$ as

$$F(x) = \begin{cases} 
-2hx^2 + hx + x & \text{if } x \leq 0.5 \\
2hx^2 - 3hx + x + h & \text{otherwise};
\end{cases}$$

and $G(z,x)$ as

$$G(z,x) = \begin{cases} 
-4h(z^2 + x^2) + 2hz + 2z & \text{if } z + x \leq 0.5 \\
8hwx - 2hz - 4hx + 2z + h & \text{otherwise.}
\end{cases}$$

Using the above low-degree polynomial formulas, it is computationally feasible to calculate the profits for any $(\alpha, z)$ combination via numerical integration. This means that for each problem instance (characterized by $V, t, h, c, d$), the monopolist’s optimal strategy may be found via a grid search over the $(\alpha, z) \in [0,1] \times [0,0.5]$ space. We implement the grid search employing a grid spacing of .001 for both $\alpha$ and $z$. In other words, for each problem instance in the problem space specified in section 4.4.1, we evaluate the profit function of the monopolist at about half a million points in its domain and select the strategy that corresponds to the maximum value derived this way.

For any $-1 \leq h \leq 1$, substituting Equation 14 into Equation 3 results in a polynomial of order $3d + 1$. (The maximum power of $\alpha$ is $d + 1$ while the maximum power of $x$ and $z$ totals to $2d$.) These polynomials can be computed very quickly for any specific value of $d$. Moreover, since we have restricted our attention to 11 values of $d$, the polynomials can easily be pre-wired to further accelerate the computations. As a result, the brute force evaluation of the profit function across the entire grid specified above is perfectly feasible for the monopolist. Using our technique, for any combination of $t > V > 0$, $-1 \leq h \leq 1$, $c > 0$, $d \in \mathbb{N}^+$, the search completes in about .01 seconds on a normal PC. Hence the total run time required for this analysis is about three hours.
Duopoly

Our assumption that $z_1 \geq z_2$ means that for each problem instance, in the grid search for Nash equilibria we only need to consider scenarios wherein we have $z_1 \geq 0.25$. Our grid-based equilibrium search extends our method to find the monopoly optima the following way. For any admissible $(\alpha_1, z_1)$, we search for the best admissible response $(\alpha_2, z_2)$ and verify if $(\alpha_1, \alpha_2, z_1, z_2)$ is indeed an equilibrium. To save on computing resources, we reduce the granularity of the $\alpha, z$ grid to 0.005 in both dimensions.

Unfortunately, brute force is no longer feasible for finding the equilibria in a duopoly. First, while the demand components (specified in Appendix C) are also simple polynomials of $\alpha_1, \alpha_2, z_1, z_2,$ and $x$, some of these polynomials are of order $4d + 2$. Second, for each admissible $(\alpha_1, z_1)$ action by Firm 1, we need to first find the admissible action $(\alpha_2, z_2)$ that maximizes Firm 2’s profits, and then check whether any alternative action by Firm 1 would increase their profit in response to $(\alpha_2, z_2)$. On a normal computer, the computation time for each combination of $V, t, c, d$ and $h$ varies from about half an hour to more than one day (longer times occur for larger values of $d$). Thus, the total computation time required for the entire parameter space is about 10,000 days on a single computer.

To circumvent this problem, we exploit that the main computational overhead occurs when computing Equation 7. Since for each $(\alpha_1, \alpha_2, z_1, z_2)$ grid, the demand function $Q_1(\cdot)$ depends only on $d$ and $h$, we can pre-compute Equation 7 for each combination of $\alpha_1, \alpha_2, z_1, z_2, d$ and $h$, and use these pre-computed grids to substitute values into Equation 8. Using this strategy, the computation on the parameter space described above finishes within two weeks using 200 nodes of a powerful computer cluster.
Appendix C - Duopoly Demand Components

Let \( W(z, x) = F(z+x) - F(z-x) \) and denote \( \overline{a}_i = 1 - \alpha_i \). Since \( F[1 - \phi] = 1 - F[\phi] \) for \( \phi \in [0, 1] \), the terms \( B_1-B_4 \) arise as

\[
B_1 = 2 \cdot \alpha_1 \cdot z^*; \\
B_2 = 2 \cdot \alpha_1 \cdot \overline{a}_2 \int \left\{ [1 - \alpha_2 \cdot \overline{a}_1] \cdot (F[z^* + x] - F[0.5 - z_2 + x]) + [1 - \alpha_2 \cdot \overline{a}_1] \cdot (F[x - 0.5 + z_2] - F[x - z^*]) + \\
\overline{a}_2 \cdot (F[(0.5 - z^*) + (0.5 - x)] + F[x - z^*] + (F[0.5 - z_2 + x] - F[x - (0.5 - z_2)]) \right\} d(x) dx \\
= 2 \cdot \alpha_1 \cdot \overline{a}_2 \int \left\{ \overline{a}_1 \cdot \alpha_2 \cdot W(x, 0.5 - z_2) - \alpha_1 \cdot \alpha_2 \cdot W(x, z^*) + \overline{a}_2 \right\} d(x) dx; \\
B_3 = 2 \cdot \overline{a}_1 \cdot \int \left\{ 1 - [1 - \alpha_1 \cdot (F[z^* + x] + F[z^* - x])] - \alpha_1 \cdot \overline{a}_2 \cdot (F[z_1 + x] - F[z^* + x]) - \alpha_1 \cdot \overline{a}_2 \cdot (F[z_1 - x] - F[z^* - x]) \right\} d(x) dx \\
= 2 \cdot \overline{a}_1 \cdot \int \left\{ 1 - [\overline{a}_2 \cdot G(z_1, x) + \alpha_2 \cdot G(z^*, x)] \right\} d(x) dx; \\
B_4 = 2 \cdot \overline{a}_1 \cdot \overline{a}_2 \int \left\{ \overline{a}_2 \cdot (F[x - z^*] + F[1 - x - z^*]) + [1 - \alpha_2 \cdot \overline{a}_1] \cdot (F[x - (0.5 - z_2)] - F[x - z^*]) + \\
[1 - \alpha_2 \cdot \overline{a}_1] \cdot (F[0.5 - x + z_2] - F[1 - x - z^*]) + (F[x + (0.5 - z_2)] - F[x - (0.5 - z_2)]) \right\} d(x) dx - \\
\int \left\{ \overline{a}_2 \cdot (F[1 - x - z_1] - F[z_1 - x]) + \overline{a}_1 \cdot \overline{a}_2 \cdot (F[x - z^*] + F[z_1 - x]) + \overline{a}_1 \cdot \overline{a}_2 \cdot (F[0.5 - x + z_2] - F[1 - x - z^*]) + \\
\overline{a}_1 \cdot \overline{a}_2 \cdot (F[x + z_1] - F[x + z^*]) + \overline{a}_1 \cdot \overline{a}_2 \cdot (F[x - (0.5 - z_2)] - F[x - z^*]) + \overline{a}_1 \cdot (F[0.5 - z_2 + x] - F[x - (0.5 - z_2)]) \right\} d(x) dx \]
\[
= 2 \cdot \overline{a}_1 \cdot \overline{a}_2 \int \left\{ \overline{a}_2 + \alpha_2 \cdot (\alpha_1 \cdot W(x, z^*) + \overline{a}_1 \cdot W(x, 0.5 - z_2)) \right\} d(x) dx - \\
\int \left\{ \overline{a}_2 + \alpha_1 \cdot G(z_1, x) + \alpha_2 \cdot W(x, 0.5 - z_2) \right\} d(x) dx \right\} .
\]