(Reverse) Price Discrimination with Information Design*

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Abstract

A monopolistic seller is marketing an indivisible good to a potential customer whose willingness to pay is determined by both his private type (personal taste) and a vertical component (quality of the good). The seller can design a menu of both prices and experiments—that reveal information about quality. We show that the optimal mechanism features both price discrimination and information discrimination: buyers with higher private types face lower prices and receive less precise positive signals. Our mechanism remains optimal within a general class of mechanisms satisfying ex post individually rationality, including those with signal-contingent prices. We consider an extension with endogenous quality (ala Mussa and Rosen, 1978) and show that the effect of information discrimination on prices is robust: higher types are provided with higher quality goods but do not necessarily face higher prices.

Keywords: price discrimination, mechanism design, information design

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1 Introduction

Price discrimination is widely studied within microeconomics, industrial organization, and price theory. To engage in price discrimination, the seller will typically either (i) charge its customers a price based on customer-specific information (first-degree price discrimination), (ii) offer a pricing scheme, usually non-linear, that induces customers to differentiate themselves based on the quantity of the good they choose to buy (second-degree price discrimination), or (iii) segment customers by some observable characteristics and charge different prices to different segments (third-degree price discrimination). In this paper, we explore another channel through which a seller may engage in price discrimination: information design.

More specifically, we analyze a model in which a seller attempts to sell a good to a buyer whose value for the good depends on both his private type (e.g., personal taste) as well as a vertical component (e.g., the quality of the good). The seller can offer a menu of prices and experiments, where experiments reveal information to the buyer about the good’s quality.

When the seller can both set prices and design information, what is the optimal mechanism? How does price discrimination interact with information design? We start by considering a simple class of direct mechanisms that work as follows. The buyer first reports his type, based on which the seller offers him a price and an experiment. After observing the result of the experiment, the buyer decides whether or not to make the purchase at the offered price. Throughout the paper, we assume that type and state are independent random variables.

We provide a complete characterization of the optimal mechanism, which has three salient features:

1. It involves both price discrimination and information discrimination. That is, different prices and different experiments are offered to buyers with different private valuations (or types).

2. Buyers with higher private valuations are offered lower prices. That is, the pricing schedule in the optimal mechanism is a decreasing function of the buyer’s type.

3. Information is disclosed through (stochastic) recommendations: each buyer type learns whether quality is above some threshold, and the threshold decreases with type. Thus, higher types are recommended to buy more frequently.

We refer to the second feature as “reverse” price discrimination, which may be surprising at first sight. To understand this feature, note that for any given price, lower types do not want the object unless the quality is relatively high. Thus, in order to convince them to buy with positive probability, the seller has to reveal a sufficiently positive signal to these buyers indicative of high quality. On the other hand, buyers with higher personal taste are willing to buy even if quality is mediocre—their priority is to find lower prices. In other words, lower types care more about information indicative of high quality, while higher types care more about prices. To maximize her expected profit, the seller tailors the optimal
mechanism to these preferences, offering more informative positive signals to lower types and lower prices to higher types.\footnote{To be sure, the experiments in the optimal mechanism are \textit{not} Blackwell-ranked: the positive signal to lower types is more precise about high quality, while the negative signal to higher types is more precise about low quality. Due to their Blackwell-incomparability, reverse price discrimination is not simply because lower types are paying higher prices for \textit{more} information, as they do not get more information in the Blackwell sense. A finer intuition based on incentive compatibility is explained at the end of Section 4.3.}

A key point we wish to convey is how price discrimination activates the role for \textit{information discrimination} (i.e., providing different information to different types). If price discrimination is not allowed, the results from Kolotilin et al. (2017) imply that the ability to design a menu of experiments does not increase the seller’s profit; that is, there is an optimal mechanism that discloses the \textit{same} information to \textit{all} types. Conversely, if information design is not allowed, then there is no scope for price discrimination within our model; the optimal mechanism is a fixed (type-independent) price. Our results therefore suggest that price discrimination and information discrimination are complimentary: once either form of discrimination can be utilized, the other is more useful and the optimal mechanism features both price discrimination and information discrimination.

The class of mechanisms we study is a natural formalization of many selling environments in real life. They also satisfy a desirable property: ex post individual rationality (ex post IR). This property requires that the buyer must have a nonnegative payoff after each type \textit{and} signal realization. In many instances, even after information disclosure, the buyer has the option to walk away without paying anything. Consumer protection policies in many countries effectively guarantee ex post IR.\footnote{For example, the European directive 2011/83/EU governs distance sales (e.g. internet and mail orders) to customer’s in the European Union. The directive grants a withdrawal right of two weeks to consumers. When a consumer exercises his withdrawal right, all contractual obligations are terminated, and the seller is required to refund all payments that have been received from the consumer. See Krähmer and Strausz (2015) for details of this policy.} Such policies rule out some of the optimal mechanisms found in the literature that require nonrefundable upfront transfers (Courty and Li, 2000; Eső and Szentes, 2007; Li and Shi, 2017). Notice that all mechanisms in the class we have considered are ex post IR, because the buyer can always choose not to buy and pay nothing after each signal realization. But if ex post IR is of our main interests, then in principle, one can consider a more general class of sequential screening mechanisms which work in two stages, as in Courty and Li (2000) and Li and Shi (2017). In Stage 1, the buyer reports his type and receives a signal realization from some type-dependent signal structure. In Stage 2, the buyer reports the signal he received, and depending on the reported type and signal, a transfer is made and the buyer gets the object with certain probability. Though sequential screening mechanisms with stochastic selling are usually not easy to analyze (see Courty and Li, 2000), we prove in Theorem 3 that the mechanism we derived is an optimal ex post IR mechanism in this general class.

Another generalization concerns the dependence of price on signal realizations. In some settings, signal realizations may be (at least partially) observable to the seller, in which case the price can be contingent on its realization. To give one example, if the salesperson is in the car during a test drive, then the price offered to the customer could vary with the result of the test drive. Theorem 2 shows that
contractible signals do not help the seller; the mechanism we derived remains optimal even if signals are observable and contractible.

Finally, we extend the model to allow for endogenous quality, where for each type of buyer, the seller can offer a combination of price, average quality and an experiment that may reveal additional information about product quality. Though the optimal pricing schedule becomes more complicated, we prove that the negative force due to information disclosure still exists. As a result, even though higher types are offered greater average quality, they do not necessarily pay higher prices.

Though our model is stylized and abstracts from the institutional details of any specific marketplace, we believe there are many applications in which the seller can control both prices and the information available to its customers. One specific example is software sales. Many software companies offer free trials of their products programmed with a convenient upgrade option within the software that pops up after the trial expires; meanwhile, they also provide the option to purchase directly from their websites without trials, often at a discounted price, which is consistent with the predictions of our model. To give an anecdotal example, as of October 2018, if searching “McAfee” on Google, one can be easily directed to a McAfee’s website that offers $25 discount off the $59.99 retail price for the “McAfee Total Protection” product. In contrast, if one first downloads the 30-day free trial version and later chooses to upgrade it to the paid version using the (conveniently located) “Upgrade Product” button inside the software, the full price will be offered.

Available relevant empirical evidence, though limited, also seems to be consistent with our model predictions. Gallaugher and Wang (1999) and Cheng and Liu (2012) document the prevalence of free trials offered by wireless carriers, softwares companies, digital TV providers, etc., and they find that compared to those without free trials, firms are able to charge higher prices from customers after the free trials. Datta, Foubert and Van Heerde (2015) find that those customers who choose to do the free trials rather than buying right away have lower retention rates. This is aligned with our prediction that lower types need information about high quality and buy less often (because the trial’s result sometimes turns out to be negative).

The rest of the paper is organized as follows. Section 2 presents the baseline model and several preliminary results. In Section 3, we solve a simple example with binary types and product qualities that helps to provide intuition for our main results. In Section 4, we derive the optimal mechanism for the model with a continuum of types and product qualities. Section 5 extends our results to a more general class of mechanisms and utility functions. Section 6 extends the model to allow the seller to choose average quality at a cost. Section 7 concludes. All proofs are in the Appendix.

Related Literature. Classical papers on optimal price discrimination include, among others, Mussa and Rosen (1978), Myerson (1981), and Armstrong, Cowan and Vickers (1995). By analyzing a natural class of ex post IR selling mechanisms with information disclosure, this paper highlights the interaction between price and information discrimination. The optimal mechanism features both reverse price
discrimination and discriminatory information disclosure, and is robust to a number of generalizations.

The results in this paper are reminiscent of the classical “quantity discount” results from Maskin and Riley (1984), if one interprets buying probability as quantity. However, in the standard monopoly pricing model with private information, the “quantity discount” arises only if the buyer’s utility is strictly concave in quantity. In contrast, when there is an indivisible single-unit object, quantity (probability of buying) enters linearly into the buyer’s expected utility. As a result, in the classical model for selling a single-unit object, the optimal mechanism features a type-independent posted price. In our results indicate that if the seller can reveal information about another component of the buyer’s willingness to pay, then (reverse) price discrimination should be used even in the sale of a single-unit object.

This paper contributes to a body of literature on mechanism design with information disclosure. In particular, Eső and Szentes (2007) and Li and Shi (2017) study optimal mechanisms under an interim IR constraint; in effect, the seller can first charge for information, and then charge for the object. When type and state are independent, both papers show that an optimal mechanism fully discloses the state to all types of buyers, and charges a type-dependent nonrefundable upfront fee and a premium conditional on buying. Our paper illustrates that, if the consumer has the option to walk away without paying after information is disclosed (i.e., ex post IR), the seller will withhold some information and disclose different information to different types (see Section 5.2 for further discussion). Smolin (2017) studies the optimal pricing and information disclosure when the good has multiple attributes and the buyer has private information about his value for each attribute. He shows that if the buyer only cares about only one of the attributes, then an optimal mechanism features a posted price and nondiscriminatory disclosure (i.e., there is neither price nor information discrimination). In the case most related to our paper where the good has only a single attribute, his result relies on the joint assumptions of multiplicative utility and zero production cost, whereas our findings apply to more general preferences and costs (see Section 5.3 for further discussion).

Bergemann, Bonatti and Smolin (2018) study the sale of information by an information provider to a private informed buyer who needs additional information for better decision making. The optimal mechanism in their paper also features price discrimination and discriminatory information disclosure, though the applications of their model are different from ours. The seller in their model does not care about the final action taken by the decision maker, while in our model the seller’s only source of profit comes from the buyer’s final purchase. In other words, their seller can only charge for information

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3 In classical models such as Maskin and Riley (1984), if (i) the object is divisible, (ii) the willingness to pay is linear in quantity, and (iii) the production cost is strictly convex in quantity, then the unit price in the optimal mechanism is strictly increasing in type.

4 The seller in our model can disclose different information to different buyer types, which differentiates our work from the classic disclosure problem in auctions with affiliated values (Milgrom and Weber, 1982; Ottaviani and Prat, 2001).

5 Hoffmann and Inderst (2011) show that once information disclosure is costly, then the seller does not want to fully disclose the state either. In our paper, however, partial disclosure is optimal even though information disclosure is costless. Li and Shi (2017) further demonstrate that if type and state are correlated (and if the seller can charge for information), discriminatory information disclosure can be optimal. In an auction setup, Bergemann and Wambach (2015) study a model where the auctioneer fully reveals the state to different bidders at different points in time or never reveals anything.
while our seller can only charge for the action (i.e., buying). This makes our model applicable to the sale of a good with the design of free trials as a useful tool, while their model is more pertinent to the sale of information.

This paper is also closely related to the literature on dynamic screening, which study the situations where a buyer receives private information in multiple stages. Courty and Li (2000) first introduce the two-stage sequential screening model, and characterize the optimal interim IR mechanism. Such a mechanism charges a type-dependent upfront fee and provides partial refund conditional on not buying. Krähmer and Strausz (2015) study the optimal ex post IR mechanism in the same framework, and show that it can be implemented by a posted price under certain assumption on the cross-hazard rate functions. In the same sequential screening framework, Bergemann, Castro and Weintraub (2017) establish a necessary and sufficient condition under which the static contract is optimal. Our paper differs because of the information design component: that is, the information revealed in the second stage is at the control of the seller, rather than exogenous. In addition to discriminatory information disclosure, our optimal mechanism has a posted price that varies with the buyer’s first-stage type but is independent of the second-stage signal. See also Bergemann and Said (2011) and Pavan (2017) for excellent surveys of recent developments in the dynamic mechanism design literature.

Furthermore, this paper is also related to the persuasion literature started by Rayo and Segal (2010) and Kamenica and Gentzkow (2011). Perhaps the most closely related is Kolotilin et al. (2017) whose model is equivalent to the information disclosure problem in our model with a fixed price and additively separable utility. They show that the seller can attain the maximum profit by public persuasion (i.e. disclosing same information to all types). An immediate corollary is that even if the seller can optimize over prices but cannot price discriminate, it is sufficient to use public persuasion to maximize profit. Our results demonstrate that once price discrimination is allowed, the seller will exploit both tools, offering different prices and information to different types of buyers in the optimal mechanism.

Finally, in a bargaining setup with private information and exogenous learning, Ning (2018) proves a result related to our finding of reverse price discrimination: in equilibrium, the buyer with a higher outside option (i.e., lower personal taste) learns for shorter time and is offered a bigger price discount.

2 The Model and Preliminaries

2.1 Setup

Payoffs

A seller (“S”, she) wants to sell a single-unit good to a buyer (“B”, he). The buyer’s valuation of the good is determined by two components: \( \theta \in \Theta \subset \mathbb{R} \) and \( \omega \in \Omega \subset \mathbb{R} \), where \( \Theta \) and \( \Omega \) are bounded. We interpret \( \theta \) as the buyer’s personal taste (private type) and \( \omega \) as the good’s quality (state of the world).

6 Here, interim IR requires that the buyer has a nonnegative payoff after knowing his type in the first stage.
The buyer’s action $a \in \{0, 1\}$ determines whether he buys the good. Given the realizations of $\theta$ and $\omega$, a price $p$ and a purchase decision $a$, the payoffs of the seller and the buyer are:

\[
\begin{align*}
  u^S &= a(p - c), \\
  u^B &= a(\theta + \omega - p),
\end{align*}
\]

where $c$ is the production cost. We assume that the seller does not derive utility from retaining the good, and that the buyer’s payoff is quasilinear in money. The additive separability between $\theta$ and $\omega$ is assumed primarily for expository purposes; in Section 5.3 we extend our results to more general class of valuation functions, including the multiplicative case.

**Information Environment**

Both $\omega$ and $\theta$ are random variables. Let $F$ and $G$ be the distribution functions of $\omega$ and $\theta$, respectively. Let $\mu = \mathbb{E}(\omega)$. Throughout the paper, we assume that $\theta$ and $\omega$ are independent. The buyer can observe his private type $\theta$, while the seller can design signal structures to reveal information about $\omega$. A signal structure, denoted by $(S, \sigma)$, consists of a signal space $S$ and a function $\sigma : \Omega \to \Delta S$. The quality $\omega$ is not per se observable to any party; only the signal realization can be observed. In particular, the seller does not have private information about $\omega$, though she can control what information to be revealed about it through her design of experiments.\(^7\)

**Mechanisms, Timing, and the Seller’s Program**

The seller can design a menu of prices and signal structures to maximize her expected profit. Specifically, a direct mechanism, $\{p(\theta), (S_\theta, \sigma_\theta)\}_{\theta \in \Theta}$, consists of a collection of prices and signal structures that vary with the buyer’s report of his type. We assume here that price is only a function of reported type, and cannot depend on signal realization. This assumption can be justified either because signals are not observable to the seller, or for contractual or legal reasons signals are not contractible. In Section 5.1, we relax this assumption and show that the mechanism we find remains optimal.

The timing of the game is as follows.

1) The seller commits a direct mechanism $\{p(\theta), (S_\theta, \sigma_\theta)\}$;

2) The buyer privately observes $\theta$, and makes a report $\hat{\theta} \in \Theta$ to the seller;

3) A signal $s$ is realized according to $(S_{\hat{\theta}}, \sigma_{\hat{\theta}})$;

4) Given $\{\theta, p(\hat{\theta}), s\}$, the buyer decides whether or not to make the purchase.

\(^7\)Throughout the paper, we retain the assumption that the seller has no private information when designing the mechanism, which helps us avoid the informed principal problem. However, in Section 6 we consider an extension where the seller can control the average quality at a cost.
Given a direct mechanism, let $E(\omega|s, \hat{\theta})$ be the posterior mean of $\omega$ when $s$ is realized from signal structure $(S_{\hat{\theta}}, \sigma_{\hat{\theta}})$. We emphasize that $\hat{\theta}$ in the conditional expectation is only used to denote the reported type; the random variables $\theta$ and $\omega$ are independent throughout this paper.

### 2.2 The Seller’s Problem

The buyer will report his type truthfully if and only if for all $\theta, \hat{\theta} \in \Theta$

$$
E_{s \sim \sigma_{\theta}} \left[ \max \left\{ 0, \theta + E(\omega|s, \theta) - p(\theta) \right\} \right] \geq E_{s \sim \sigma_{\theta}} \left[ \max \left\{ 0, \theta + E(\omega|s, \hat{\theta}) - p(\hat{\theta}) \right\} \right].
$$

(II-0)

On both sides of this inequality, the buyer optimizes his purchase decision after receiving the report-dependent price and a signal from the report-dependent signal structure. We say that a direct mechanism is **incentive compatible (IC)** if it satisfies (II-0).

The seller’s program is

$$
\max_{(p(\cdot), (S, \sigma))} \mathbb{E}_{\theta} \left[ (p(\theta) - c) \Pr(\text{type } \theta \text{ buys}) \right]
$$

s.t. (II-0)

where

$$
\Pr(\text{type } \theta \text{ buys}) = \sigma_{\theta}\left( \{ s : \theta + E(\omega|s, \theta) - p(\theta) \geq 0 \} \right).
$$

**Sufficiency of Recommendation Mechanisms**

A **recommendation mechanism** is one such that $S_{\theta} = \{0, 1\}$ for all $\theta \in \Theta$ and

$$
\theta + E(\omega|s = 1, \theta) - p(\theta) \geq 0, \ \text{whenever } \sigma_{\theta}(s = 1) > 0;
$$

$$
\theta + E(\omega|s = 0, \theta) - p(\theta) \leq 0, \ \text{whenever } \sigma_{\theta}(s = 0) > 0.
$$

That is, it is a direct mechanism in which information disclosure is through obedient recommendations: a realized recommendation reveals information about $\omega$ and the buyer finds it optimal to take the recommended action. The following lemma implies that it is without loss to focus on recommendation mechanisms.

**Lemma 1.** For any IC direct mechanism, there exists an IC recommendation mechanism that generates the same profit to the seller.

Intuitively, for any IC direct mechanism, we can construct an IC recommendation mechanism with the same pricing function which generates the same behavior. In particular, the original signal structure is garbled in such a way that for any reported type and signal realization, $(\hat{\theta}, s)$, that induces buying (not
buying), the new signal is $\tilde{s} = 1$ ($\tilde{s} = 0$). Under this mechanism, the buyer, if truthfully reporting, finds it optimal to take the recommended action (i.e. it is a recommendation mechanism) and achieves the same interim payoff (after knowing his type) as before; moreover, since all signal structures become less informative, the interim payoff from misreporting is weakly lower than before. Because the buyer does not want to misreport his type in the original mechanism, neither does he in the new recommendation mechanism.

A recommendation mechanism can be described by $\{p(\theta), q(\omega, \theta)\}$, where $q(\omega, \theta)$ is the probability of $s = 1$ (recommending to buy) when the state is $\omega$ and the report is $\theta$. The IC constraint for recommendation mechanisms can be decomposed into two parts. First, conditional on reporting truthfully, the buyer should always be willing to obey the recommended action. Second, the buyer should want to report truthfully.

**Obedience**

Take any recommendation mechanism $\{p(\theta), q(\omega, \theta)\}$. If type $\theta$ reports truthfully, he will be willing to obey a recommendation to buy if

$$\theta + \mathbb{E}(\omega|s = 1, \theta) - p(\theta) = \frac{\int_{\Omega} [\theta + \omega - p(\theta)]q(\omega, \theta)dF(\omega)}{\int_{\Omega} q(\omega, \theta)dF(\omega)} \geq 0. \quad (1)$$

Similarly, he will be willing to obey a recommendation not to buy if

$$\theta + \mathbb{E}(\omega|s = 0, \theta) - p(\theta) = \frac{\int_{\Omega} [\theta + \omega - p(\theta)](1 - q(\omega, \theta))dF(\omega)}{\int_{\Omega} [1 - q(\omega, \theta)]dF(\omega)} \leq 0. \quad (2)$$

It can be shown that (1) and (2) are satisfied if and only if

$$V(\theta) \equiv \int_{\Omega} [\theta + \omega - p(\theta)]q(\omega, \theta)dF(\omega) \geq \max\{0, \theta + \mu - p(\theta)\}, \quad (3)$$

where the LHS is the buyer’s interim expected payoff after knowing his type and reporting truthfully (assuming that he obeys the recommendation).

To see why (3) is necessary and sufficient for obedience, observe that there are three ways to disobey a recommendation mechanism:

(i) never buying regardless of the recommendation;
(ii) always buying regardless of the recommendation;
(iii) buying when recommended not to buy, and not buying when recommended to buy.

\[\text{In a general signal structure } (S_\theta, \sigma_\theta), \sigma_\theta(\omega) \text{ is a distribution over } S_\theta \text{ for each } \omega. \text{ Since now the signal space is } \{0, 1\}, \text{ a distribution } \sigma_\theta(\omega) \text{ over } \{0, 1\} \text{ is fully characterized by the probability of sending signal 1, which we denote by } q(\omega, \theta).\]
The two terms inside the max operator on the RHS represent the payoffs of (i) and (ii). And (iii) cannot improve the buyer’s payoff unless either (i) or (ii) does.

**Truthful Reporting**

If type $\theta$ reports $\hat{\theta}$ and later obeys the recommendation, his interim payoff is

$$U(\theta, \hat{\theta}) = \int_\Omega [\theta + \omega - p(\hat{\theta})]q(\omega, \hat{\theta})dF(\omega). \quad (4)$$

When considering the buyer’s incentive to misreport his type, we must also take into account the possibility of “double deviations”; that is, the buyer first lies about his type, and then disobeys the recommendation. It turns out that misreporting type and always disobeying the recommendation is never optimal provided that (3) holds.$^9$ Therefore, truthful reporting requires:

$$V(\theta) \geq \max_{\hat{\theta} \in \Theta} \left\{ 0, \theta + \mu - p(\hat{\theta}), U(\theta, \hat{\theta}) \right\}, \text{ for all } \theta \in \Theta. \quad (IC)$$

Notice that (IC) implies (3) and therefore fully summarizes the IC constraint for recommendation mechanisms. Hence, we can write the seller’s program as

$$\max_{\{p(\theta), q(\omega, \theta)\}} \mathbb{E}_\theta \left[ (p(\theta) - c) \int_\Omega q(\omega, \theta)dF(\omega) \right]$$

s.t. (IC)

where $\int_\Omega q(\omega, \theta)dF(\omega)$ is the probability of type $\theta$ being recommended to buy.

## 3 The Model with Binary States and Types

In this section, we consider a special case of the model with binary states and types to illustrate the complementary nature of information discrimination and price discrimination. In particular, we demonstrate

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$^9$ To see this, suppose that type $\theta$ reports $\hat{\theta}$. If $\theta > \hat{\theta}$, we have

$$\int_\Omega (\theta + \omega - p(\hat{\theta}))q(\omega, \hat{\theta})dF(\omega) > \int_\Omega (\hat{\theta} + \omega - p(\hat{\theta}))q(\omega, \hat{\theta})dF(\omega) \geq 0$$

where the weakly inequality follows from the obedience of type $\hat{\theta}$. So it is strictly optimal for type $\theta$ follow the “buy” recommendation after reporting $\hat{\theta}$.

If $\theta < \hat{\theta}$, we have

$$\int_\Omega (\theta + \omega - p(\hat{\theta}))(1 - q(\omega, \hat{\theta}))dF(\omega) < \int_\Omega (\hat{\theta} + \omega - p(\hat{\theta}))(1 - q(\omega, \hat{\theta}))dF(\omega) \leq 0$$

where the weakly inequality again follows from the obedience of type $\hat{\theta}$. So it is strictly optimal for type $\theta$ follow the “don’t buy” recommendation after reporting $\hat{\theta}$.
that price discrimination activates a role for information discrimination and both are part of the optimal mechanism. We will also highlight several key features of the optimal mechanism that will carry over to the model with a continuum of states and types.

Consider the following special case of our setup:

- \( \Omega = \{0, 1\} \), where \( \mu = \Pr(\omega = 1) \)
- \( \Theta = \{\theta_L, \theta_H\} \), where \( \lambda = \Pr(\theta = \theta_H) \)

**Benchmark 1: No Persuasion**

Consider first a benchmark without information design in which the seller can only offer a menu of prices and quantities \( \{p(\theta), x(\theta)\}_{\theta \in \{\theta_L, \theta_H\}} \), where \( x(\theta) \) corresponds to the probability that a buyer who reports type \( \theta \) is offered the good at price \( p(\theta) \). In this case, the classic result of Myerson (1981) applies and the optimal mechanism can be implemented with a single fixed price (either \( \mu + \theta_L \) or \( \mu + \theta_H \)).

**Benchmark 2: Uniform Pricing**

As a second benchmark, consider the case in which the seller cannot price discriminate (i.e., \( p(\theta_L) = p(\theta_H) \)), but she can reveal different information to different types. In this case, a result similar to the one in Kolotilin et al. (2017) obtains: information discrimination is not beneficial for the seller.\(^{10}\)

**Proposition 1** (Kolotilin et al. (2017)). *Suppose that the seller is restricted to charge a uniform price (i.e., \( p(\theta_L) = p(\theta_H) = p \)). Then, the seller cannot benefit from information discrimination.*

In other words, without the ability to price discriminate, it is without loss to consider mechanisms in which the seller provides the same experiment to all buyer types.\(^{11}\)

**Price Discrimination with Information Design**

We now describe the optimal mechanism with both price discrimination and information design. In Appendix A.5, we characterize the optimal mechanism in the binary model for all parameter values. To simplify exposition and avoid trivial cases, for the remainder of this section we normalize \( c = 0 \) and focus on parameters satisfying the following assumption.

**Assumption 1.** \( 0 = \theta_L < \theta_H < \mu \) and \( \lambda \leq \frac{\mu - \theta_H}{\mu} \).

Let \( p^* \) denote the optimal price when the seller is restricted to uniform pricing (Benchmark 2). Under Assumption 1, in the optimal mechanism \( p^* = \mu \) and no information is disclosed, both buyers

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\(^{10}\) Kolotilin et al. (2017) analyze a model with a continuum of types so our method of proof differs from theirs.

\(^{11}\) If one is interested in whether the optimal mechanism can be implemented by providing the same experiment to all buyer types, then it is not without loss of generality to restrict attention to recommendation mechanisms.
buy with probability 1, and the seller extracts full surplus (equal to \(\mu\)) from the low type. The question is how to best extract more surplus from the high type when price discrimination is allowed. Let us consider a perturbation of the optimal mechanism with uniform pricing. Suppose that the seller charges a higher price \(p_L > p^*\) to the low type, and uses the “Bayesian Persuasion” disclosure policy. That is, the signal structure provided to the low type generates a distribution of posterior beliefs, a random variable denoted by \(\nu = \Pr(\omega = 1)\), whose support is \([0, p_L]\). The seller’s profit obtained from type L is \(p_L \times \Pr(\nu = p_L) = \mu\), i.e., same as when the price is nondiscriminatory.

Before specifying the price and signal structure for type H, let us first draw type H’s payoff graphically. Given a price \(p\) and a posterior belief \(\nu\) about the probability of \(\omega = 1\), type H’s utility (after optimizing over purchase decision) is

\[
\max\{\theta_H + \nu - p, 0\}.
\]

In Figure 1, the black curve depicts type H’s indirectly utility as a function of his belief given the price \(p^* = \mu\). In the optimal mechanism with uniform pricing, since type H buys with probability 1, his expected utility is \(\theta_H + \mu - p^*\). Now consider type H’s payoff if he chooses type L’s contract in the perturbed mechanism. The red curve in Figure 1 draws the utility of type H when he reports \(\theta_L\) and faces the price \(p_L\). As we just mentioned, the random posterior induced by the signal structure after reporting \(\theta_L\) has support \([0, p_L]\). Notice that type H’s utility when \(\nu = p_L\) is \(\theta_H + \nu - p_L = \theta_H\), which is same as his utility in the optimal mechanism with uniform pricing; meanwhile, his utility when \(\nu = 0\) is 0, which is strictly less than \(\theta_H\). This implies that type H’s expected utility from misreporting is strictly less than his expected utility in the optimal mechanism with uniform pricing. In other words, type H’s incentive constraint is relaxed under the modified contract for type L.

As a result, if now type H’s contract involves a higher price than before, i.e., \(p_H > p^*\), while still recommending him to buy with probability one, type H is willing to report truthfully and always buy. In particular, \(p_H\) can be chosen as high as what the blue curve in Figure 1 illustrates, in which case type H is indifferent between the two reports. Since type H is charged a higher price and he always buys, the seller’s profit from type H is higher than before.\(^{12}\)

In this perturbed mechanism, we have argued that the profit from type L is same as before while that from type H is strictly higher, so we conclude that this particular perturbation strictly increase the seller’s expected profit. Hence, if the seller can engage in price discrimination and information design, he will do both and will provide different information to different types.

The optimal mechanism exploits the above perturbation to its extreme. That is, the seller increases the price for type L all the way to the highest possible price, which is 1.

**Proposition 2.** Suppose that Assumption 1 holds. Then the optimal menu in the binary model involves

\(^{12}\) Type L would not want to deviate to type H’s contract either, because \(p_H\) is higher than type L’s willingness to pay \(\mu\), so after misreporting type L will never buy. This gives him 0 payoff, the same as his expected payoff when reporting truthfully.
\[ \theta_H + \nu - p \]

optimal mechanism with uniform pricing

perturbed contract for \( \theta_H \)

perturbed contract for \( \theta_L \)

\[ \theta_H + \mu - p^* \]

\[ \theta_H + \mu - p_H \]

\[ 0 \quad p^* - \theta_H \quad p_H - \theta_H \quad \mu \quad p_L - \theta_H \quad p_L \quad 1 \]

\[ \nu \]

Figure 1: Type H’s Indirect Utility as a Function of Posterior Beliefs

- \( p^*_L = 1 \) and the buyer is recommended to buy if and only if quality is high (i.e., \( \omega \) is fully revealed to type L).

- \( p^*_H = \mu + (1 - \mu)\theta_H < 1 \) and the buyer is always recommended to buy (i.e., no information about \( \omega \) is revealed to type H).

Similar to the perturbed contract, in the optimal mechanism the seller uses the “Bayesian Persuasion” disclosure policy for type L. Further, the seller charges the high type a price such that he is indifferent between reports and is always recommended to buy.

As we will show in the next section, three key features of this mechanism carry over to the model with a continuum of types and states. Namely,

1. The seller employs both price and information discrimination,

2. Higher types are offered lower prices, and

3. When recommended to buy, lower types have higher expectations about quality \( \omega \).

Intuitively, in order to convince those buyers with lower ex ante valuations, the seller must let her “buy” recommendation be more indicative of high quality; meanwhile, buyers with higher personal taste are willing to buy even if the expected quality is mediocre, but they care more about shopping at a lower price. The seller tailors the optimal mechanism in such a way that higher types are recommended to buy more frequently, but conditional on a “buy” recommendation, lower types have higher posterior perceptions about quality and face a higher price.
4 The Model with a Continuum of States and Types

We now analyze the general model with a continuum of states and types. Suppose that \( \Theta = [\underline{\theta}, \bar{\theta}] \) and \( \Omega = [\omega, \bar{\omega}] \), that \( G \) has a strictly positive density functions \( g \) on \( \Theta \), and that \( F \) has a strictly positive density functions \( f \) on \( \Omega \). To avoid trivial cases, we assume that \( \underline{\theta} + \omega \leq c < \bar{\theta} + \bar{\omega} \). In addition, we employ the following regularity condition.

**Assumption 2 (Monotone Hazard Rate).** \( \frac{g(\theta)}{1 - G(\theta)} \) is strictly increasing on \( \Theta \).

Assumption 2 is frequently used in the mechanism design literature and is satisfied by many commonly used distributions such as uniform and (truncated) normal distributions.

From equation (4), we have

\[
U(\theta, \hat{\theta}) = A(\hat{\theta}) + \theta B(\hat{\theta}),
\]

where

\[
A(\hat{\theta}) = \int_{\omega}^{\bar{\omega}} (\omega - p(\hat{\theta}))q(\omega, \hat{\theta})dF(\omega),
\]

\[
B(\hat{\theta}) = \int_{\omega}^{\bar{\omega}} q(\omega, \hat{\theta})dF(\omega).
\]

Above, \( B(\hat{\theta}) \) is the unconditional probability of a buyer being recommended to buy after he reports \( \hat{\theta} \). The linearity of \( U \) in \( \theta \) is due to our assumption on the additive separability between type and quality, which we will relax later.

We will solve the seller’s problem as follows. First, we relax (IC) to

\[
V(\theta) \geq U(\theta, \hat{\theta}), \quad \forall \theta, \hat{\theta} \in \Theta, \tag{IC'}
\]

and use the standard mechanism design approach to solve the seller’s problem subject to (IC’). We then prove that the solution to the relaxed program satisfies the original constraint (IC).

4.1 The Seller’s Relaxed Problem

Consider the following program:

\[
\max_{\{p(\theta), q(\omega, \theta)\}} \int_{\theta}^{\hat{\theta}} (p(\theta) - c)B(\theta)dG(\theta), \tag{6}
\]

s.t. (IC’)

Electronic copy available at: https://ssrn.com/abstract=3263898
Lemma 2. A recommendation mechanism satisfies \((IC')\) if and only if

\[
V(\theta) = V + \int_{\theta}^{\theta} B(s) ds; \quad \text{and}
\]

\(B(\theta)\) is nondecreasing.

Lemma 2 is a standard envelope-theorem characterization of IC when the buyer’s utility is quasilinear in money. By Lemma 2, we have

\[
V + \int_{\theta}^{\theta} B(s) ds = V(\theta) = A(\theta) + \theta B(\theta),
\]

which implies

\[
p(\theta)B(\theta) = -V + \theta B(\theta) - \int_{\theta}^{\theta} B(s) ds + \int_{\omega}^{\omega} \omega q(\omega, \theta) dF(\omega)
\]  

(7)

So the seller’s relaxed program can be written as

\[
\max_{\{q(\omega, \theta), V\}} - V + \int_{\theta}^{\theta} \left[(\theta - c)B(\theta) - \int_{\theta}^{\theta} B(s) ds + \int_{\omega}^{\omega} \omega q(\omega, \theta) dF(\omega)\right] dG(\theta)
\]

s.t. \(B(\theta)\) is nondecreasing and \(V \geq 0\).

By setting \(V = 0\), integrating the second term by parts, and noting that \(B(\theta) = \int_{\omega}^{\omega} q(\omega, \theta) dF(\omega)\), the objective function in program (8) can be written as

\[
\int_{\theta}^{\theta} \int_{\omega}^{\omega} (\omega - m(\theta)) q(\omega, \theta) g(\theta) dF(\omega) d\theta,
\]

(9)

where \(m(\theta)\) is the additive inverse of “virtual” surplus, i.e.,

\[
m(\theta) \equiv -\left(\theta - \frac{1 - G(\theta)}{g(\theta)} - c\right).
\]

Pointwise optimization yields the following candidate solution to the relaxed program (8):

\[
q^*(\omega, \theta) \equiv \begin{cases} 
1, & \text{if } \omega \geq m(\theta) \\
0, & \text{if } \omega < m(\theta)
\end{cases}
\]

(10)

That is, any type \(\theta\) is recommended to buy if and only if the quality is above a threshold, \(m(\theta)\). Furthermore, the threshold is decreasing in type (by Assumption 2). As a result, higher types are recommended to buy more frequently.

Under this candidate solution, each type is recommended to buy with unconditional probability

\[
B^*(\theta) = 1 - F(m(\theta)).
\]

(11)
Since \( m(\theta) \) is decreasing, \( B^*(\theta) \) satisfies the nondecreasing constraint, so \( q^* \) is indeed a solution to the relaxed program (8).

Given our candidate allocation rule, the price needed to satisfy (IC') follows almost immediately from (7). In particular,

\[
p^*(\theta) = \theta + \mathbb{E}(\omega|\omega \geq m(\theta)) - \int_{\theta_1}^{\theta} \frac{B^*(s)\,ds}{B^*(\theta)}, \quad \forall \theta \in [\theta_1, \theta],
\]

where \( \theta_1 \equiv \inf_\Theta \{ \theta : B^*(\theta) > 0 \} \) denotes the lowest type that is recommended to buy with positive probability under \( q^* \). The term \( \theta + \mathbb{E}(\omega|\omega \geq m(\theta)) \) in equation (12) is type \( \theta \)'s posterior expected utility from purchasing when he receives the recommendation to buy; therefore, \( \int_{\theta_1}^{\theta} B^*(s)\,ds \) is the information rent of type \( \theta \).

If \( \theta_1 = \theta \), then (12) applies to all \( \theta \in \Theta \). If \( \theta_1 > \theta \), then a positive measure of the lowest types are never recommended to buy and the price is not uniquely pinned down. For our purposes, it will suffice to set

\[
p^*(\theta) = \theta_1 + \bar{\omega}, \quad \forall \theta \in [\theta, \theta_1).
\]

We can correspondingly define \( \theta_2 \equiv \sup_\Theta \{ \theta : B^*(\theta) < 1 \} \), so any type between \( \theta_1 \) and \( \theta_2 \) buys with probability strictly between 0 and 1. Our assumption, \( \bar{\theta} + \omega \leq c < \bar{\theta} + \bar{\omega} \), ensures that \( \theta_1 \) and \( \theta_2 \) are well defined with \( \bar{\theta} \leq \theta_1 < \theta_2 \leq \bar{\theta} \).

### 4.2 The Seller’s Original Program

We now return to the seller’s original program (5). We will argue that \( \{p^*, q^*\} \) defined above satisfies (IC) and therefore is a solution to the seller’s original program (5). Before doing so, we discuss a novel and important property of the mechanism.

**Proposition 3.** Suppose that Assumption 2 holds. Then \( p^*(\theta) \) is decreasing in \( \theta \) for all \( \theta \in \Theta \) and strictly decreasing on \((\theta_1, \theta_2)\).

Proposition 3 states that a lower price is charged to a higher type, which may look surprising at the first sight. Nevertheless, the key here is that this price is charged only when the buyer makes the purchase, and that the buyer’s willingness to pay depends both on his type and the information revealed about \( \omega \). From equation (10), we can see that the threshold of \( \omega \) for the “buy” recommendation decreases with \( \theta \). As a result, when a low type is recommended to buy (which is a rare event), his posterior expectation of \( \omega \) is very high; meanwhile, when a high type is recommended to buy, his

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13 When the buyer’s type is observable to the seller, for each \( \theta \), the optimal price charged is \( \theta + \mathbb{E}(\omega|\omega \geq c - \theta) \) and the buyer is recommended to buy whenever \( \omega \geq c - \theta \). Unsurprisingly, the seller extracts the entire surplus.
posterior expectation about $\omega$ is not far from $\mu$. Thus, as $\theta$ increases, the two components of the buyer’s valuation move in opposite directions; it turns out that in the candidate solution, the negative force from the movement of $\mathbb{E}(\omega|s = 1, \theta)$ dominates, and the price decreases with type.

Proposition 3 is also useful for solving the seller’s original problem. Recall that the actual IC constraint faced by the seller is $V(\theta) \geq \max_{\hat{\theta} \in \Theta} \left\{ 0, \theta + \mu - p(\hat{\theta}), U(\theta, \hat{\theta}) \right\}$, for all $\theta \in \Theta$.

Since $p^*$ is decreasing, for each type $\theta$ we only need to additionally check $\theta + \mu - p^*(\hat{\theta})$; that is, the deviation to reporting $\hat{\theta}$, and then always buying regardless of recommended actions. It turns out that, for each type $\theta$, after reporting a higher type (say, $\bar{\theta}$), it is never optimal to buy when recommended against so. This implies that, for any type who reports $\bar{\theta}$, ignoring the recommendation and always buying is dominated by always following the recommendation. Since we already know that the latter is not profitable, we conclude that any double deviation is not profitable either and that $\{p^*, q^*\}$ satisfies the original IC constraint (IC).

**Theorem 1.** Suppose that Assumption 2 holds. Then $\{p^*, q^*\}$ is an optimal recommendation mechanism. In this mechanism, a higher type buys more often and pays a lower price conditional on buying. Moreover, if $\{\tilde{p}, \tilde{q}\}$ is another optimal recommendation mechanism, then $\tilde{p}(\theta) = p^*(\theta)$ for all $\theta > \theta_1$.

Though our Assumption 2 on monotone hazard rate is common in the literature and holds for many frequently used distributions, we should emphasize the dual role it plays in our analysis. First, like in standard mechanism design problems, it ensures that the virtual surplus is increasing in buyer’s type. Second, it guarantees that no buyer wants to “double-deviate” by misreporting his type and always buying, because it generates reverse price discrimination.

Relaxing the monotone hazard rate assumption (while still assuming monotone virtual surplus) complicates the optimal pricing schedule, and in some cases, renders our current solution approach invalid. For example, in the case where $\bar{\theta}$ is high enough such that $\bar{\theta} + \omega - c \geq 0$, our solution approach is valid if and only if $\min_{\theta \in \Theta} p^*(\theta) = p(\bar{\theta})$; that is, in the solution to the relaxed program (6), the highest type pays the lowest price, whereas $p^*$ is not necessarily decreasing once the monotone hazard rate assumption is relaxed. In fact, under only the monotone virtual surplus condition, one can construct examples in which the price schedule in the solution to the relaxed program is everywhere increasing.

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14 One can show that the obedience constraint (3) is satisfied by the candidate mechanism $\{p^*, q^*\}$. (The proof of this step uses Assumption 2). Then, if type $\bar{\theta}$’s payoff from purchasing is less than 0 when he is recommended not to buy, the same is true for any type below it after reporting $\bar{\theta}$.

15 It is necessary because, when $\bar{\theta} + \omega - c \geq 0$, the highest type $\bar{\theta}$ buys with probability 1 in the candidate mechanism, and type $\theta$ must be charged the lowest price to ensure his truthful reporting. It is sufficient because any buyer contemplating the double deviation (where he first misreports his type and then always buys) will report the highest type $\bar{\theta}$; as type $\theta$ is recommended to buy with probability 1, “reporting $\bar{\theta}$ and then always buying” is equivalent to “reporting $\bar{\theta}$ and then always following the recommendation”, but we already know that the latter is not profitable.
and thus such a candidate solution violates the IC constraint \((IC)\). We leave a formal investigation of the optimal mechanism in these irregular cases for future research.

4.3 Discriminatory Information Disclosure and Reverse Price Discrimination

In this subsection, we examine the information disclosed in the optimal mechanism, and explain the economic intuition behind our mechanism.

Kolotilin et al. (2017) study essentially our model with a fixed price, and they show that discriminatory information disclosure is not needed. An immediate corollary of their result is that, if the seller in our model cannot price discriminate, then it suffices to use public information disclosure, i.e., disclosing same information to all types. In contrast, we have shown that once price discrimination is allowed, then she will exploit both tools, offering different prices and information to different buyers.

To better illustrate, we depict the quality threshold for each buyer type above which he is recommended to buy, and compare it with two benchmarks. In particular, the black curve in Figure 2 draws the threshold quality \(m(\theta)\) in the optimal mechanism. The blue curve draws the efficiency threshold: there are gains from trade for type \(\theta\) if and only if \(\omega > c - \theta\). Finally, if we fix the price \(p^*(\theta)\) for type \(\theta\), but let the buyer perfectly observe the state, he will buy if the realized quality satisfies \(\omega > \theta - p^*(\theta)\); this threshold is depicted by the red curve.\(^{16}\) As we can see from Figure 2, in the optimal mechanism, each buyer type buys less often than efficient trade but more often than when he can perfectly observe the state and faces the same price.

The following proposition formalizes what Figure 2 illustrates.

**Proposition 4.** \(c - \theta < m(\theta) < p^*(\theta) - \theta\) for all \(\theta \in (\theta_1, \theta_2)\).

![Figure 2: Quality Thresholds for Buying with \(c = 0\), \(\Omega = [-1, 1]\), \(\Theta = [-1, 2]\) and Uniform Distributions](https://ssrn.com/abstract=3263898)

\(^{16}\) Such a menu is not incentive compatible (so it is not of interest per se), but the comparison of these thresholds is useful for explaining the intuition of our solution.
Proposition 4 is useful for understanding why reverse price discrimination is necessary for incentive compatibility in the optimal mechanism. Let us first think about what features are reasonable for optimal information disclosure to have, and then what they imply about prices. A profit-maximizing seller would like to sell to each type of buyer with high probability. This leads to two features of information disclosure in the optimal menu.

1. Each type is recommended to buy when the quality is sufficiently high, and the object is sold to higher types with greater probabilities (i.e. the quality thresholds for higher types are lower).

2. Fixing a type and the price he is offered, the buyer is recommended to buy more often than if \( \omega \) was fully disclosed, as Proposition 4 suggests.

The first feature is not surprising, because one insight we have from standard mechanism design problems is that optimal allocation, though usually socially inefficient, should at least track efficient allocation. Efficiency requires that each type of buyer gets the object when quality is high enough, while the thresholds for higher types are lower; this very much resembles the first feature of information disclosure described above. The second feature is due to the following reasoning. Fix a type and the price to that type. If quality (state) is fully revealed, the buyer will buy if and only if the realized quality is above some threshold, and this threshold is the buyer’s most preferred one (because he cannot be made better off than having full information about quality). In a recommendation mechanism, the seller has control over the threshold subject to obedience. Intuitively, the seller will choose a threshold lower than the buyer’s preferred one, because even if the good states are combined with some bad ones, the buyer would still find it optimal to follow the recommendation while he is induced to buy more often, thus generating a higher expected profit.

In Figure 2, these two features are reflected in the thresholds depicted by the black curve being (i) decreasing and (ii) always below the red curve.\(^{17}\) Given these features of optimal disclosure, let us think about prices. Fix any type \( \theta \in (\theta_1, \theta_2) \). If price does not change with type, the buyer would want to report a (slightly) lower type than his true one. This is because the threshold quality for type \( \theta \) under optimal disclosure (black in Figure 2) is too low compared to type-\( \theta \)’s most preferred threshold (red in Figure 2), and reporting a lower type can increase the threshold he gets. So if price does not vary with type, then each type of buyer has an incentive to report a lower type in order to get a more valuable recommendation rule for himself. As a result, to induce truthful reporting in the optimal mechanism, a lower type must be offered a higher price.

\(^{17}\) One may notice that the second feature is explained using some intuition from the case where there is only one type. When there are multiple types, the incentive compatibility constraint also comes into play. Our assumption on monotone hazard rate ensures that such an intuition still holds for every type.
5 Generalizations

5.1 Contractible Signals

In the model we considered, the price can only vary with the buyer’s (reported) type. But one can think of situations where price can depend on signal realizations. For example, after letting the customer test drive a car, the car dealer can, in principle, charge different prices depending on the result of the test drive. We will show that even if price can be a function of signal realizations, the mechanism described in Theorem 1 remains optimal.

A direct mechanism with contractible signals \( \{p(\theta, s), (S_\theta, \sigma_\theta)\} \) consists of a signal structure \((S_\theta, \sigma_\theta)\) for each \(\theta\), and a pricing function \(p : \Theta \times S_\theta \to \mathbb{R}_+\) which offers a price that depends on both the reported type \(\theta\) and the realized signal \(s\) from \((S_\theta, \sigma_\theta)\). A recommendation mechanism with contractible signals is one such that \(S_\theta = \{0, 1\}\) for all \(\theta \in \Theta\) and

\[
\begin{align*}
\theta + \mathbb{E}(\omega|s = 1, \theta) - p(\theta, s = 1) &\geq 0, \text{ whenever } \sigma_\theta(s = 1) > 0; \\
\theta + \mathbb{E}(\omega|s = 0, \theta) - p(\theta, s = 0) &\leq 0, \text{ whenever } \sigma_\theta(s = 0) > 0.
\end{align*}
\]

Similar to Lemma 1, the following lemma shows that in this case, it is without loss to focus on recommendation mechanisms with contractible signals.

**Lemma 3.** For any IC direct mechanism with contractible signals, there exists an IC recommendation mechanism with contractible signals that generates the same profit to the seller.

We can describe a recommendation mechanism with contractible signals by \(\{p(\theta, 1), p(\theta, 0), q(\omega, \theta)\}\), where \(q(\omega, \theta)\) is the probability of \(s = 1\) (recommending to buy) when the state is \(\omega\) and the report is \(\theta\), and \(p(\theta, 1)\) \((p(\theta, 0))\) is the price when he is recommended (not) to buy. Notice that \(p(\theta, 0)\) is somewhat redundant because this price is not actually charged and is only used to ensure that the buyer does not want to buy when receiving \(s = 0\). Essentially, for a mechanism to be a recommendation mechanism, we now only need to ensure that the buyer is willing to follow the positive recommendation; that is

\[
\int_{\Omega} [\theta + \omega - p(\theta, 1)] q(\omega, \theta) dF(\omega) \geq 0.
\]  \hspace{1cm} (14)

When receiving \(s = 0\), that \(\int_{\Omega} [\theta + \omega - p(\theta, 0)](1 - q(\omega, \theta)) dF(\omega) < 0\) is always satisfied when \(p(\theta, 0)\) is very large for all \(\theta\) (greater than a buyer’s highest possible willingness to pay).

Moreover, if \(p(\theta, 0)\) is very large for all \(\theta\), we never need to worry about the type of double deviations in which type \(\theta\) reports \(\hat{\theta}\) and then buys when recommended against so. Therefore, we can write
the IC constraint for recommendation mechanisms with contractible signals as
\[
\int_\Omega \left[ \theta + \omega - p(\theta, 1) \right] q(\omega, \theta) dF(\omega) \geq \max \left\{ 0, \int_\Omega \left[ \theta + \omega - p(\hat{\theta}, 1) \right] q(\omega, \hat{\theta}) dF(\omega) \right\}, \forall \theta, \hat{\theta} \in \Theta.
\]
\[(\text{IC-contractible})\]

Since \(p(\theta, 0)\) is already set to a very large number without affecting incentive constraints, the seller’s program is
\[
\max_{\{p(\theta, 1), q(\omega, \theta)\}} \mathbb{E}_\theta \left[ (p(\theta, 1) - c) \int_\Omega q(\omega, \theta) dF(\omega) \right].
\]
\[(15)\]
\[\text{s.t. (IC-contractible)}\]

Comparing programs (15) and (5), we can see that (IC-contractible) relaxes (IC). But in Section 4, the constraint (IC’) in program (6) is more relaxed than both (IC-contractible) and (IC); that is,
\[
\text{(IC)} \Rightarrow \text{(IC-contractible)} \Rightarrow \text{(IC’)}.
\]

Since we have shown in Theorem 1 that the solution to program (6) under the most relaxed constraint (IC’) solves program (5) under the most stringent constraint (IC), this implies that the same solution solves the current program (15) with (IC-contractible), as well.

\textbf{Theorem 2.} Suppose that Assumption 2 holds. The mechanism described in Theorem 1 remains optimal even if signals are contractible.

Theorem 2 demonstrates that the ability to let the price depend on signal realizations does not benefit the seller.

\subsection*{5.2 General Ex Post Individually Rational Mechanisms}

One can define ex post individual rationality (ex post IR) by requiring that the buyer must have a non-negative payoff after every realization of type and signal. All (direct) mechanisms we have considered so far are ex post IR, because the buyer pays nothing when he does not buy, and can always optimize his purchase decision conditional on his type, signal realization and price.

In principle, we can consider a more general class of mechanisms as in Eső and Szentes (2007) and Li and Shi (2017). In this subsection, we show that the optimal mechanism characterized in Theorem 1 is an optimal ex post IR mechanism within this more general class of mechanisms.

\textbf{Unobservable Signals}

Suppose that signal realizations are not observable to the seller. As in Li and Shi (2017), we say that a general mechanism \(\{(S_\theta, \sigma_\theta), (X(\theta, s), T(\theta, s))\}_{\theta \in \Theta}\) consists of a signal structure \((S_\theta, \sigma_\theta)\) and a selling mechanism \((X(\theta, s), T(\theta, s))\) for each type, where \(X(\theta, s)\) and \(T(\theta, s)\) are the trading probability
and transfer given a reported type $\theta$ and a reported signal realization $s$. A general mechanism works in two stages. In stage one, the buyer reports his type and receives a signal realization from $(S_\theta, \sigma_\theta)$; in stage two, the buyer reports his signal, and given his reported type and signal, the buyer pays $T(\theta, s)$ and gets the object with probability $X(\theta, s)$.

For a general mechanism $\{(S_\theta, \sigma_\theta), (X(\theta, s), T(\theta, s))\}$, we say that it is incentive compatible if

$$\mathbb{E}_{s \sim \sigma_\theta} [X(\theta, s)(\theta + \mathbb{E}(\omega|s)) - T(\theta, s)] \geq \mathbb{E}_{s \sim \sigma_{\hat{\theta}}} \left[ \max_{\hat{s} \in S_{\hat{\theta}}} \left[ X(\hat{\theta}, \hat{s})(\theta + \mathbb{E}(\omega|s)) - T(\hat{\theta}, \hat{s}) \right] \right], \forall \theta, \hat{\theta} \in \Theta.$$  

(IC-general)

That is, truthfully reporting one’s type and received signal is weakly better than any deviation that could involve lying about both type and signal. In addition, we say that it is ex post IR if it satisfies

$$X(\theta, s)(\theta + \mathbb{E}(\omega|s)) - T(\theta, s) \geq 0, \text{ for all } \theta \in \Theta \text{ and } s \in S_\theta.$$  

(ex post IR)

**Contractible Signals**

Similarly, we can define a general mechanism with contractible signals $\{(S_\theta, \sigma_\theta), (X(\theta, s), T(\theta, s))\}_{\theta \in \Theta}$ by a signal structure and a selling mechanism for each type $\theta$, with the difference being that $X(\theta, s)$ and $T(\theta, s)$ are now functions of the realized signal (instead of the reported signal), and the buyer only needs to report his type.

For a general mechanism with contractible signals $\{(S_\theta, \sigma_\theta), (X(\theta, s), T(\theta, s))\}$, it is incentive compatible if it satisfies

$$\mathbb{E}_{s \sim \sigma_\theta} [X(\theta, s)(\theta + \mathbb{E}(\omega|s)) - T(\theta, s)] \geq \mathbb{E}_{s \sim \sigma_{\hat{\theta}}} \left[ X(\hat{\theta}, \hat{s})(\theta + \mathbb{E}(\omega|s)) - T(\hat{\theta}, \hat{s}) \right], \forall \theta, \hat{\theta} \in \Theta.$$  

(IC-general-contractible)

The difference between (IC-general-contractible) and (IC-general) is that, on the RHS, the buyer can no longer optimize over signal reports because signals are now (observable and) contractible.

**Theorem 3.** Suppose that Assumption 2 holds. The mechanism described in Theorem 1 is an optimal ex post IR general mechanism, regardless of whether or not signals are contractible.

Note that, in the optimal mechanism, the selling probability conditional on $\theta$ and $s$ is either 1 or 0. Analogous to the well-known result on the sufficiency of posted price in the sale of a good to a privately informed buyer, Theorem 3 suggests that when the seller can reveal information about an independent variable, a mechanism that uses type-dependent posted prices is optimal.

**Relation to the Existing Results**

Eső and Szentes (2007) study the general mechanisms with interim IR constraint (instead of ex post IR)
and binary information disclosure: the seller chooses between fully revealing the state and revealing nothing. Interim IR requires that the buyer’s expected utility after knowing his private type must be nonnegative, while allowing for negative utility after some signal realizations. They show that if $\theta$ and $\omega$ are independent, an optimal mechanism has the following structure, regardless of whether or not signal realizations are contractible:

- full disclosure of $\omega$ to all types; and
- a nonrefundable entry fee $c(\theta)$ and a price $p(\theta)$ for each type.

In contrast, if one requires the mechanism to satisfy ex post IR, Theorem 3 shows the seller will withhold information (instead of full disclosure) and provide different information to different types. In other words, an optimal ex post IR mechanism features both price and information discrimination.

These differences in results also differentiate our model applications from theirs. Their model is more about how to first optimally charge for (full) information via the upfront fee and then optimally charge for the good, while our model captures the sale of a good with information design as a useful tool.

### 5.3 General Valuation Functions

In this subsection, we consider the general valuation function form $u(\theta, \omega)$. We provide bounds on the cross partial derivative of the valuation function such that the optimal mechanism takes exactly the same form as in Theorem 1.

Let us retain the setup in Section 2, except that now the buyer’s utility function is

$$u^B = a(u(\theta, \omega) - p),$$

where $u : \Theta \times \Omega \to \mathbb{R}$ is twice continuously differentiable with $u_\theta, u_\omega > 0, u_{\theta\theta}, u_{\omega\omega} \leq 0$ on $\Theta \times \Omega$. We will focus on the model with a continuum of states and types. As in Section 4, we assume $u(\theta, \omega) \leq c < u(\bar{\theta}, \bar{\omega}).$

**Theorem 4.** Suppose that Assumption 2 holds. There exist $\bar{M} > 0$ and $\underline{M} < 0$, such that if $\bar{M} < u_{\theta\omega}(\theta, \omega) < \underline{M}$ for all $\theta \in \Theta$ and $\omega \in \Omega$, then an optimal mechanism has the same structure as that in Theorem 1, regardless of whether or not signal realizations are contractible.

In Theorem 4, the upper bound ensures that the unconstrained recommendation rule takes the cutoff form with higher types buying more often. The lower bound preserves the “reverse price discrimination” feature of the solution to the relaxed program (Proposition 3), which is sufficient to guarantee that the solution to the relaxed program satisfies (IC) (as discussed in Section 4.2).

---

18 The nonrefundable entry fee violates ex post IR because with positive probability a buyer pays this fee and later finds it optimal not to buy after receiving the signal, in which case the buyer’s ex post payoff is $-c(\theta) < 0.$

19 Li and Shi (2017) shows that discriminatory information disclosure can be optimal if $\theta$ and $\omega$ are correlated.
A multiplicative valuation function \( u(\theta, \omega) = \theta \times \omega \) and a strictly positive production cost \( (c > 0) \) satisfies all the conditions of Theorem 4, and thus enjoys the same characterization. While his primary focus differs from ours, Smolin (2017) considers the case with (i) multiplicative valuation function and (ii) zero production cost and shows that a revenue-maximizing mechanism involves a posted price with no information disclosure to any type. This result relies on both (i) and (ii). Once the cost is strictly positive, our Theorem 4 shows that (reverse) price discrimination and discriminatory information disclosure are needed for profit maximization. Indeed, the upper bound \( \overline{M} \) in Theorem 4 (see equation (27) in Appendix A.3 for the precise expression) varies with model primitives including the production cost \( c \). With a multiplicative valuation function, \( \overline{M} = 1 \) (\( \overline{M} > 1 \)) whenever \( c = 0 \) \( (c > 0) \) and thus the upper bound in Theorem 4 is satisfied if and only if \( c > 0 \). Meanwhile, our analysis shows that if the valuation function is, for example, additive instead of multiplicative, then our characterization holds even with zero production cost.

6 Extension: Quality Design

In some situations, in addition to controlling information disclosure about quality, the seller can influence quality directly. For example, by designing different packages or levels of services, the seller may be able to control the (average) quality of the product, while at the same time further disclosing information about the (realized) quality via free trials.

In this section, we extend our model to allow the seller to choose an average quality for each type of buyer at a cost. Under a multiplicative valuation function, we will show that optimal quality choice satisfies the same first-order condition as in the classical model of Mussa and Rosen (1978), while optimal information disclosure has the same cutoff structure as in Section 3 of this paper. With respect to pricing, the negative effect from information disclosure still exists: though higher types are offered greater average quality, they may face lower prices, so that optimal pricing schedule can be non-monotone in buyer’s type.

6.1 Setup

We keep most of the setup in Section 5.1 with the following modifications.

Payoffs

Let \( \theta \in [0, \overline{\theta}] \subset \mathbb{R}_+ \) be the buyer’s private type, and \( \omega \) be the quality of the product, satisfying

\[
\omega = \mu + \epsilon,
\]

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where \( \mu \in [\underline{\mu}, \infty) \in \mathbb{R}_+ \) is the average quality which the seller can control, and \( \epsilon \in [\underline{\epsilon}, \bar{\epsilon}] \) is a noise term, independent of \( \theta \) and \( \mu \), about which the seller can design experiments to reveal information. We assume that \( \mu + \epsilon \geq 0 \), so that the realized quality is always nonnegative.

Given the realizations of \( \theta \) and \( \epsilon \), an average quality \( \mu \), a price \( p \) and a purchase decision \( a \), the payoffs of the seller and the buyer are:

\[
\begin{align*}
    u^S &= a[p - c(\mu)], \\
    u^B &= a[\theta(\mu + \epsilon) - p],
\end{align*}
\]

where \( c(\mu) \) is the cost of producing a good of average quality \( \mu \). We assume that the cost function is twice continuously differentiable, strictly increasing and strictly convex, and \( \lim_{\mu \to \infty} c'(\mu) = \infty \). Here, we adopt the multiplicative valuation function \( u(\theta, \omega) = \theta \omega \) as in Mussa and Rosen (1978); if the valuation function is additively separable in \( \theta \) and \( \omega \), there will be no quality differentiation and the characterization in Theorem 1 directly applies.

### Information and Timing

The buyer’s type \( \theta \) and quality noise \( \epsilon \) are independent random variables. Let \( F \) and \( G \) be the distribution functions of \( \epsilon \) and \( \theta \), respectively, with strictly positive densities \( f \) and \( g \). We assume that \( \mathbb{E}(\epsilon) = 0 \), and will maintain Assumption 2 that \( \frac{g(\theta)}{1 - G(\theta)} \) is increasing in \( \theta \).

We consider a mechanism design problem where the seller can offer to each buyer three things: price, average quality, and an experiment of realized quality. When designing the contract, the seller does not know the realization of the noise term in quality. Specially, the timing of the game is as follows.

1) The seller commits to a direct mechanism \( \{p(\theta), \mu(\theta), (S_\theta, \sigma_\theta)\} \);

2) The buyer privately observes \( \theta \), and makes a report \( \hat{\theta} \in \Theta \) to the seller;

3) A signal \( s \) is realized according to \( (S_{\hat{\theta}}, \sigma_{\hat{\theta}}) \);

4) Given \( \{\theta, p(\hat{\theta}), \mu(\hat{\theta}), s\} \), the buyer decides whether or not to make the purchase.

### 6.2 Analysis

#### Optimal Mechanism

The idea of Lemma 1 readily applies to this setup, so we can without loss focus on recommendation mechanisms represented by \( \{p(\theta), \mu(\theta), q(\epsilon, \theta)\} \), where \( q(\epsilon, \theta) \) is the probability of recommending the buyer to buy given report \( \theta \) and state \( \epsilon \). Similar to the baseline model, the IC constraint still needs to cover “double deviations”. As before, we solve this problem by first ignoring such deviations, and then
verify (using a more subtle argument) that the solution to such a relaxed problem satisfies the original IC constraint.

In the relaxed problem which ignores double deviations, the seller’s objective can be reduced to:

\[
\max_{\{\mu(\theta), q(\epsilon, \theta)\}} \int_0^\theta \int_\epsilon^\theta q(\epsilon, \theta) \left[ \left( \theta - \frac{1 - G(\theta)}{g(\theta)} \right) \left( \mu(\theta) + \epsilon \right) - c(\mu(\theta)) \right] g(\theta)dF(\epsilon)d\theta, \tag{16}
\]

with the pricing schedule \(p(\theta)\) satisfying

\[
p(\theta) \int_\epsilon^\theta q(\epsilon, \theta)dF(\epsilon) = D(\theta, \theta) - \int_0^\theta D_1(s, s)ds, \tag{17}
\]

where

\[
D(\theta, \hat{\theta}) = \int_\epsilon^\hat{\theta} \theta (\mu(\hat{\theta}) + \epsilon)q(\epsilon, \hat{\theta})dF(\epsilon).
\]

Pointwise maximization of (16) suggests the following candidate for quality choice and a recommendation rule. For quality choice, at each \(\theta\) such that \(\theta - 1 - G(\theta) > 0\), \(\mu^*(\theta)\) satisfies the FOC:

\[
c'(\mu^*(\theta)) = \theta - \frac{1 - G(\theta)}{g(\theta)}. \tag{18}
\]

Note that \(\mu^*(\theta)\) is increasing in \(\theta\) because \(c\) is strictly convex and the RHS of (18) is increasing in \(\theta\).

For a recommendation rule,

\[
q^*(\epsilon, \theta) = \begin{cases} 
1, & \text{if } \epsilon \geq m(\theta) \\
0, & \text{if } \epsilon < m(\theta)
\end{cases} \tag{19}
\]

where

\[
m(\theta) = \frac{c(\mu^*(\theta))}{\theta - \frac{1 - G(\theta)}{g(\theta)}} - \mu^*(\theta).
\]

One can show that the threshold is decreasing in \(\theta\), as in the baseline model, higher types are recommended to buy more often.

Turning now to the price, equation (17) implies that

\[
p^*(\theta) = \mathbb{E} [\theta(\mu(\theta) + \epsilon) | \epsilon \geq m(\theta)] - \frac{\int_0^\theta D_1^*(s, s)ds}{\Pr(\text{recommending } \theta \text{ to buy})}. \tag{20}
\]

In the baseline model, the verification of the original IC constraint is greatly simplified due to the
“reverse price discrimination” feature. With endogenous quality, the price is not necessarily decreasing in type. Nevertheless, by a more subtle argument, we can still show that the solution from the relaxed program satisfies the original IC constraint.

**Theorem 5.** Suppose that Assumption 2 holds. An optimal recommendation mechanism exists and has the following structure. There exist \( \theta_1 > 0 \) such that

- For \( 0 \leq \theta \leq \theta_1 \), the buyer is never recommended to buy;
- For \( \theta > \theta_1 \), the quality offered is
  
  \[
  \mu^*(\theta) = \max \left\{ \mu, c'^{-1} \left( \theta - \frac{1 - G(\theta)}{g(\theta)} \right) \right\},
  \]

  the price offered is
  
  \[
  p^*(\theta) = \mathbb{E}[\theta(\mu(\theta) + \epsilon) | \epsilon \geq m(\theta)] - \frac{\int_0^\theta D_1^*(s, s) ds}{\Pr(\text{recommending } \theta \text{ to buy})},
  \]

  and the buyer is recommended to buy whenever \( \epsilon \geq m(\theta) \).

In this mechanism, higher types are offered higher average quality and buy more often, but do not necessarily pay a higher price.

**Non-monotone Pricing**

Due to the variation in average quality across buyers, the pricing schedule varies in a more complicated way than that in the baseline model. Nevertheless, we can prove the following proposition.

**Proposition 5.** \( p^*(\theta) = \theta \mu^*(\theta) + I(\theta) \), where \( \mu^*(\theta) \geq 0 \) and \( I(\theta) < 0 \) for all \( \theta > \theta_1 \). As a result, \( p^*(\theta) \) can be non-monotone in \( \theta \).

Proposition 5 decomposes the total variation of price with buyer’s type into two parts, one capturing the positive effect from quality improvement as in Mussa and Rosen (1978), and the other capturing the negative effect from information disclosure as in the baseline model.\(^{21}\) As a result of these opposing forces, the price can vary non-monotonically with type.

Figure 3 illustrates how the non-monotonicity can arise. In both panels, for a region of types just above \( \theta_1 \), the optimal average quality is the minimum quality. Since quality does not vary with type in that region, price is decreasing as in the baseline model. As buyer’s type further increases, the positive

---

21 To be precise, the change of price with type reflects the variations of four elements: type, average quality, information disclosure, and information rent. Without quality choice, Proposition 3 shows that the total effect of the remaining three is negative; with quality choice, we can show that the total effect from those three elements (i.e., type, information disclosure, and information rent) on prices is still negative, while the effect from quality improvement is positive.
effect from quality improvement dominates, so that price starts increasing; but for sufficiently high
types, it is possible (as shown in the right panel) that the negative effect from information disclosure
becomes dominating, and price decreases again with type.

\[ p^*(\theta) \]

\[ \theta \]

\[ \bar{\theta} \]

Figure 3: Optimal Pricing Schedules Under Different Noise Distributions

6.3 Comparison with Mussa and Rosen (1978)

The model studied in this section can be viewed as an extension of Mussa and Rosen (1978) to include
information design about quality noise. Of course, the seller can choose not to further disclose any
information about quality, in which case optimal pricing and quality choice are given by the solution
to their classical model. When the seller can offer and design free trials for its customers, how does it
affect optimal pricing? Where does the extra profit due to information disclosure come from? Here we
provide partial answers to these questions.

In our model, the FOC (18) for quality choice remains the same as in Mussa and Rosen (1978). However, the seller’s profit still changes through two channels: extensive and intensive margins. First,
by offering free trials, the seller is able to sell with positive probabilities to more types of buyers, which
increases her profit through extensive margin. This happens to those buyers with low ex ante valuations
whose (virtual) surplus is negative at the average quality, but is positive when the noise term is large; in
other words, the seller can offer those types a free trial whose positive result is very indicative of high
quality, so when such a result comes out, the buyer is willing to make the purchase that he would not
make without doing the trial. Second, for those types that the seller already sold to, offering free trials
has two intuitive effects: the seller is able to charge higher prices from them conditional on buying as
they have received additional (positive) information about quality; meanwhile, they purchase with less
probability (less than 1) because trials can fail. Compared to Mussa and Rosen (1978), the change of

\[ \text{supp}(\epsilon) = [-1.5, 1.5], f(\epsilon) = 1/3 \] for the left panel, \( f(\epsilon) = \frac{\epsilon^{2.5}}{2^{2.5}+\epsilon^{2.5}} \) for the right panel; moreover, \( \theta \sim U[0, 20], c(\mu) = 7 + \mu^{2.5} \) for both panels. These parameters imply that \( \theta_1 = 11.63 \). Both panels are drawn for \( \theta \in [\theta_1, \bar{\theta}] \) where \( \bar{\theta} = 17.44 \); for \( \theta > \bar{\theta} \), the buyer buys with probability 1, price is increasing and its derivative coincides
with the price derivative in Mussa and Rosen (1978).

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profits from these types depends on how the entire mechanism changes, and is generally unclear. We provide a sufficient condition under which prices are indeed higher, and we numerically illustrate the ambiguous effect on profits through intensive margin.

Let \( p^*_\text{MR}(\theta) \) be the optimal pricing schedule when information disclosure is not allowed, and let \( \theta_{\text{MR}} \) be the lowest type which the seller sells to.\(^{23}\) As the threshold function \( m(\theta) \) is decreasing in \( \theta \), it is not hard to see that

\[
\theta_{\text{MR}} = m^{-1}(0) > m^{-1}(\bar{\epsilon}) = \theta_1.
\]

That is, without further disclosing information, the seller is unable to sell to those types between \( \theta_1 \) and \( \theta_{\text{MR}} \).

**Proposition 6.** If \( \frac{1 - G(\theta)}{g(\theta)} \) is convex in \( \theta \), then \( p^*(\theta) \geq p^*_\text{MR}(\theta) \) for all \( \theta \geq \theta_{\text{MR}} \).

Proposition 6 provides a sufficient condition under which information design allows the seller to charge higher prices to all existing customers (i.e., those types which she already sold to without information design). It is satisfied by, among others, uniform and (truncated) normal distributions. Interestingly, the condition is only on the buyer’s type distribution and puts no restriction on the cost function or the distribution of quality noise.

Compared to the case without information disclosure, the change in the seller’s profits from these types (i.e., \( \theta \geq \theta_{\text{MR}} \)) is ambiguous. As shown in Figure 4, even when all types are charged higher prices than before (left panel), the expected profits from some of these types can decrease (right panel), as they buy less often. In fact, under the uniform distribution which satisfies the assumption in Proposition 6, the expected profits from all types \( \theta \geq \theta_{\text{MR}} \) decreases. The seller is willing to sacrifice profits on the intensive margin, because the extra profit generated on the extensive margin (i.e., selling to more types) dominates.

**Figure 4:** Effects of Information Design on Prices and Expected Revenues

In summary, disclosing information about realized quality affects the seller’s profit through two

\[^{23}\text{One can show that } p^*_\text{MR}(\theta) = \theta \mu^*(\theta) - \int_{\theta_{\text{MR}}}^{\theta} \mu^*(s) \, ds, \text{ where } \mu^* \text{ is the same as in Theorem (5), and } \theta_{\text{MR}} \text{ solves } \left( \theta - \frac{1 - G(\theta)}{g(\theta)} \right) \mu^*(\theta) - c(\mu^*(\theta)) = 0.\]**
channels. On the extensive margin, the seller expands its customer base by selling to buyers whose type is between $\theta_1$ and $\theta_{MR}$ with positive probability. On the intensive margin, the seller charges higher prices, though sells with lower probability meaning that the profits can decrease. Overall, the additional profits from new customers outweigh the potential losses from existing customers.

7 Conclusion

We study the sale of a single-unit good to a buyer whose willingness to pay depends on both his personal taste and the quality of the good. We characterize the profit-maximizing selling mechanism with information disclosure about quality, and show that the optimal mechanism features “reverse” price discrimination and discriminatory information disclosure. To be convinced to make the purchase, buyers with lower ex ante valuations need more precise information about high quality; in contrast, higher types are willing to buy quality is mediocre, they put relatively more importance on lower prices. To maximize her expected profit, the seller tailors the optimal mechanism according these preferences, offering more precise positive signals to lower types and lower prices to higher types with more frequent selling.

Our results illustrate the complimentary relationship between price discrimination and information discrimination. If the seller is not allowed to price discriminate, then the ability to information discriminate does not increase her profit: she can attain the maximum profit by disclosing same information to all types. Conversely, when information discrimination is not allowed, there is no scope for price discrimination in our model: she can attain the maximum profit by posting a fixed price. Only when the seller is able to discriminate along both dimensions will it be beneficial for her to do so.

The optimality of the mechanism derived in this paper is robust. It remains optimal even when we allow for contractible signals, non-additively-separable valuation functions, and stochastic selling. In particular, it is an optimal ex post IR mechanism in the general class of sequential screening mechanisms, regardless of whether or not signals are contractible. Our findings are also robust to an extension with endogenous quality, in which case the price may be non-monotone in the buyer’s type.

Throughout our analysis, we have assumed that the buyer’s personal taste and quality are independent. If they are correlated, the approach that we have used to solve the model in this paper is no longer valid. In fact, as Li and Shi (2017) illustrate, the optimal interim IR mechanism becomes substantially different with correlation. We leave an exploration of the case with correlation as a direction for future research.
A Appendix

A.1 Proofs of Results in Sections 2 and 4

Proof of Lemma 1. Take any IC direct mechanism \((p(\theta), (S_\theta, \sigma_\theta))\). Let us construct another direct mechanism \((\tilde{p}(\theta), (\tilde{S}_\theta, \tilde{\sigma}_\theta))\) as follows. Let \(\tilde{p} = p\), \(\tilde{S}_\theta = \{0, 1\}\) for all \(\theta\); and for each \(\theta\), let \(\{\{0, 1\}, \tilde{\sigma}_\theta\}\) be a Blackwell garbling of \((S_\theta, \sigma_\theta)\) such that the signal realization \(\tilde{s}\) from \((\{0, 1\}, \tilde{\sigma}_\theta)\) is a deterministic function of the signal realization \(s\) from \((S_\theta, \sigma_\theta)\):

\[
\tilde{s}(s) = \begin{cases} 
1, & \text{if } \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \geq 0; \\
0, & \text{if } \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) < 0.
\end{cases}
\]

For each type, the new signal is realized in two steps. In the first step, a signal \(s\) is realized from the original signal structure. In the second step, if \(s\) originally induces the buyer to buy, then \(\tilde{s} = 1\); otherwise, \(\tilde{s} = 0\).

We first show that \((\tilde{p}(\theta), (\tilde{S}_\theta, \tilde{\sigma}_\theta))\) is IC. For type \(\theta\), if he reports truthfully, his expected payoff is the same as before (LHS of (IC-0)) by our construction of \(\tilde{s}\). On the other hand, if he reports \(\hat{\theta} \neq \theta\), his expected payoff is weakly less than before (RHS of (IC-0)) because \((\{0, 1\}, \tilde{\sigma}_\theta)\) is a Blackwell garbling of \((S_\theta, \sigma_\theta)\). Therefore, (IC-0) is still satisfied by the new mechanism.

Moreover, in the new mechanism,

\[
\theta + \mathbb{E}(\omega|\tilde{s} = 1, \theta) - p(\theta) = \theta + \int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \geq 0\}} \mathbb{E}(\omega|s, \theta) d\sigma_\theta(s) - p(\theta)
\]

\[
= \frac{\int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \geq 0\}} \left[ \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \right] d\sigma_\theta(s)}{\sigma_\theta(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \geq 0\})}
\]

\[
\geq 0, \text{ whenever } \tilde{\sigma}_\theta(\tilde{s} = 1) > 0;
\]

\[
\theta + \mathbb{E}(\omega|\tilde{s} = 0, \theta) - p(\theta) < 0, \text{ whenever } \tilde{\sigma}_\theta(\tilde{s} = 0) > 0.
\]

So the new mechanism is a recommendation mechanism.

Finally, the new mechanism generates the same profit because \(\tilde{p} = p\) and

\[
\Pr(\text{type } \theta \text{ buys}) = \tilde{\sigma}_\theta(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \geq 0\})
\]

\[
= \tilde{\sigma}_\theta(\tilde{s} = 1)
\]

\[
= \sigma_\theta(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta) \geq 0\}).
\]

\(\square\)

Proof of Lemma 2. This follows from standard envelope theorem argument. \(\square\)

Observation 1. Let \(x\) be a random variable with CDF \(F\) and strictly positive density \(f\) on \([x, \bar{x}]\), and let \(e(y) \equiv
$\mathbb{E}(x|x \geq y)$. Then,

$$e'(y) = \frac{f(y) \int_y^\infty (1 - F(x))dx}{(1 - F(y))^2}.$$ 

**Proof.**

$$e(y) = \int_y^\infty xf(x)dx \frac{1}{1 - F(y)} = \frac{y(1 - F(y)) + \int_y^\infty (1 - F(x))dx}{1 - F(y)} = y + \int_y^\infty (1 - F(x))dx.$$ 

Then,

$$e'(y) = 1 + \frac{-(1 - F(y))^2 + f(y) \int_y^\infty (1 - F(x))dx}{(1 - F(y))^2} = \frac{f(y) \int_y^\infty (1 - F(x))dx}{(1 - F(y))^2}.$$

**Proof of Proposition 3.** By its definition in (12) and (13), $p^*$ is constant on $[\theta_1, \theta_1]$ and $[\theta_2, \theta]$, and is continuous at $\theta_2$. For the possible case where $\theta_1 > \theta$ (that is, $m(\theta) > \bar{\omega} = m(\theta_1)$), let us first show that $p^*$ is continuous at $\theta_1$. Note that

$$\lim_{\theta_1 \downarrow \theta} p^*(\theta) = \theta_1 - \lim_{\theta_1 \downarrow \theta_1} \left[ \frac{\int_{\theta_1}^{\theta} B^*(s)ds}{B^*(\theta)} - \frac{\int_{m(\theta)}^{\bar{\omega}} \omega dF(\omega)}{B^*(\theta)} \right] = \theta_1 - \frac{B^*(\theta_1) + m(\theta_1)f(m(\theta_1))m'(\theta_1)}{B^*(\theta_1)} = \theta_1 - \frac{\bar{\omega}f(\bar{\omega})m'(\theta_1)}{f(\bar{\omega})m'(\theta)} = \theta_1 + \bar{\omega} = p(\theta_1),$$

where the second line follows from L’Hopital’s rule and the third line follows from $B^*(\theta_1) = 0$ and $m(\theta_1) = \bar{\omega}$.

Now we show that $p^*$ is decreasing on $(\theta_1, \theta_2)$. By Observation 1, for $\theta \in (\theta_1, \theta_2)$ we have

$$\frac{dp^*(\theta)}{d\theta} = 1 - \frac{B^*(\theta)^2 - B^*(\theta) \int_{\theta_1}^{\theta} B^*(s)ds}{B^*(\theta)^2} + \frac{f(m(\theta)) \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega}{(1 - F(m(\theta)))^2} m'(\theta) = \frac{B^*(\theta)}{B^*(\theta)^2} \left( \int_{\theta_1}^{\theta} B^*(s)ds - \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega \right),$$

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where the second line follows from (11) and \( B^*(\theta) = -f(m(\theta))m'(\theta) > 0 \).

In the above expression, one can see that

\[
\int_{\theta_1}^{\theta_2} B^*(s)ds - \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega \bigg|_{\theta=\theta_1} = - \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega \leq 0
\]

because \( m(\theta_1) \leq \bar{\omega} \).\(^24\) Moreover, for all \( \theta \in [\theta_1, \theta_2] \),

\[
\frac{d}{d\theta} \left( \int_{\theta_1}^{\theta} B^*(s)ds - \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega \right) = B^*(\theta) - \left[ -(1 - F(m(\theta))) \right] m'(\theta) = B^*(\theta)(1 + m'(\theta)) < 0
\]

because \( m'(\theta) < -1 \). Therefore, we have

\[
\int_{\theta_1}^{\theta} B^*(s)ds - \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega < 0, \forall \theta \in (\theta_1, \theta_2),
\]

and thus \( \frac{dp^*}{dp} < 0 \) on \( (\theta_1, \theta_2) \).

\[ \square \]

**Proof of Theorem 1.** We need to show \( \{p^*, q^*\} \) satisfies (IC). By definition, \( \{p^*, q^*\} \) satisfies (IC):

\[
V(\theta) \geq U(\theta, \hat{\theta}), \forall \theta, \hat{\theta} \in \Theta.
\]

Since \( V = 0 \), Lemma 2 implies that \( V(\theta) \geq 0 \) for all \( \theta \).

Next, we show that, for all \( \theta \in [\hat{\theta}, \bar{\theta}] \), \( \theta + E[\omega|\omega < m(\theta)] - p^*(\theta) \leq 0 \); that is, when recommended not to buy, each type finds it optimal to follow the recommendation. Note that, for \( \theta \in [\theta_1, \bar{\theta}] \),

\[
\theta + E[\omega|\omega < m(\theta)] - p^*(\theta) \leq \theta + m(\theta) - p^*(\theta) = \theta + m(\theta) - \left[ \theta + E[\omega|\omega \geq m(\theta)] - \int_{\theta_1}^{\theta} \frac{B^*(s)ds}{B^*(\theta)} \right]
\]

\[
= \frac{1}{B^*(\theta)} \left( \int_{\theta_1}^{\theta} B^*(s)ds - \int_{m(\theta)}^{\bar{\omega}} (1 - F(\omega))d\omega \right) \leq 0 \quad \text{(21)}
\]

where the last inequality follows from our analysis of the terms in the parenthesis in the proof of Proposition 3.

For \( \theta \in [\theta, \theta_1] \) if any, they are never recommended to buy, which they optimally follow because type \( \theta_1 \) does so. Up to now, we have shown that \( \{p^*, q^*\} \) satisfies the obedience constraint (3).

To prove that \( \{p^*, q^*\} \) satisfies (IC), it remains to verify that \( V(\theta) \geq \max_\theta \theta + \mu - p(\hat{\theta}) \). By Proposition 3,

\[ \text{Recall that } \hat{\theta}_1 = \inf_{\theta} \{B^*(\theta) > 0\} = \inf_{\theta} \{m(\theta) < \bar{\omega}\}. \]

33
\[
\min_{\hat{\theta}} p(\hat{\theta}) = p(\bar{\theta}).
\]
So for any \( \theta \in [\bar{\theta}, \tilde{\theta}] \), we have
\[
\max_{\hat{\theta}} \theta + \mu - p(\hat{\theta}) = \theta + \mu - p(\bar{\theta})
\]
\[
= B^*(\bar{\theta}) \left[ \theta + \mathbb{E}[\omega|\omega \geq m(\theta)] - p(\bar{\theta}) \right] + (1 - B^*(\bar{\theta})) \left[ \theta + \mathbb{E}[\omega|\omega < m(\theta)] - p(\bar{\theta}) \right]
\]
\[
\leq B^*(\bar{\theta}) \left[ \theta + \mathbb{E}[\omega|\omega \geq m(\theta)] - p(\bar{\theta}) \right] + (1 - B^*(\bar{\theta})) \left[ \bar{\theta} + \mathbb{E}[\omega|\omega < m(\theta)] - p(\bar{\theta}) \right]
\]
\[
\leq B^*(\bar{\theta}) \left[ \theta + \mathbb{E}[\omega|\omega \geq m(\theta)] - p(\bar{\theta}) \right] + (1 - B^*(\bar{\theta})) \left[ \bar{\theta} + \mathbb{E}[\omega|\omega < m(\theta)] - p(\bar{\theta}) \right]
\]
\[
= U(\theta, \bar{\theta})
\]

where the first inequality follows from \( \theta \leq \bar{\theta} \), and the second inequality follows from (21). Therefore, \((p^*, q^*)\) satisfies (IC).

To show the uniqueness of optimal pricing schedule, take any recommendation mechanism \{\tilde{p}, \tilde{q}\} that solves (5). Since \{\tilde{p}, \tilde{q}\} generates the same profit as \{p^*, q^*\} and the latter solves the relaxed program (6) (or equivalently program (8)), we know that \{\tilde{p}, \tilde{q}\} also solves program (8). From our rewriting of the objective in equation (9), \(\tilde{q}\) must satisfy that for all \( \theta \in \Theta \),
\[
\tilde{B}(\theta) = \int_{\omega} \tilde{q}(\omega, \theta) dF(\omega) = 1 - F(m(\theta)) = B^*(\theta).
\]

Lemma 2 and equation (7) then imply that \(\tilde{p}(\theta) = p^*(\theta)\) whenever \(B^*(\theta) > 0\); that is, whenever \( \theta > \theta_1 \).

**Proof of Proposition 4.** By definition, \( m(\theta) = \frac{1 - G(\theta)}{g(\theta)} - \theta + c > c - \theta \). It remains to be shown that
\[
m(\theta) < p^*(\theta) - \theta.
\]

By equation (12), we have
\[
p^*(\theta) - \theta = \mathbb{E}[\omega|\omega \geq m(\theta)] - \frac{\int_{\theta_1}^{\theta} B^*(s) ds}{B^*(\theta)}
\]
\[
= m(\theta) + \frac{\int_{m(\theta)}^{\omega_\theta} (1 - F(\omega)) d\omega}{1 - F(m(\theta))} - \frac{\int_{\theta_1}^{\theta} B^*(s) ds}{B^*(\theta)}
\]
\[
= m(\theta) + \frac{\int_{m(\theta)}^{\omega_\theta} (1 - F(\omega)) d\omega}{1 - F(m(\theta))} - \frac{\int_{\omega_\theta}^{m(\theta)} \frac{1 - F(\omega)}{m(m^{-1}(\omega))} d\omega}{1 - F(m(\theta))}
\]
\[
= m(\theta) + \frac{\int_{m(\theta)}^{\omega_\theta} (1 - F(\omega)) \left(1 - \frac{1}{m(m^{-1}(\omega))}\right) d\omega}{1 - F(m(\theta))}
\]
\[
> m(\theta),
\]

where the third line follows from (11), and the last line follows from \(-m'(\theta) > 1\) so that \(1 - \frac{1}{-m'(m^{-1}(\omega))} > 0\).
A.2 Proofs of Results in Sections 5.1 and 5.2

Proof of Lemma 3. The IC constraint for any direct mechanism with contractible signals \( \{p(\theta, s), (S_\theta, \sigma_\theta)\} \) can be written as:

\[
\mathbb{E}_{s \sim \sigma_\theta} \left[ \max \{0, \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s)\} \right] \geq \mathbb{E}_{s \sim \sigma_\theta} \left[ \max \left\{0, \theta + \mathbb{E}(\omega|s, \hat{\theta}) - p(\hat{\theta}, s)\right\} \right] \\
\text{(IC-1)}
\]

The expected profit generated by an IC mechanism with contractible signals is

\[
\int_{\Theta} \left[ \int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\}} (p(\theta, s) - c)d\sigma_\theta(s) \right] dG(\theta)
\]

where the inner integral is the expected profit from type \( \theta \) because the price \( p(\theta, s) \) is charged only when the buyer finds it optimal to buy upon receiving \( s \).

Take an IC direct mechanism with contractible signals \( \{p(\theta, s), (S_\theta, \sigma_\theta)\} \). Let us construct another direct mechanism \((\tilde{p}(\theta, s), (\tilde{S}_\theta, \tilde{\sigma}_\theta))\) as follows. Let \( \tilde{S}_\theta = \{0, 1\} \) for all \( \theta \); and for each \( \theta \), let \((\{0, 1\}, \tilde{\sigma}_\theta)\) be a Blackwell garbling of \((S_\theta, \sigma_\theta)\) such that the signal realization \( \tilde{s} \) from \((\{0, 1\}, \tilde{\sigma}_\theta)\) is a deterministic function of the signal realization \( s \) from \((S_\theta, \sigma_\theta)\):

\[
\tilde{s}(s) = \begin{cases} 1, & \text{if } \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0; \\ 0, & \text{if } \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) < 0. \end{cases}
\]

For each type, the new signal is realized in two steps. In the first step, a signal \( s \) is realized from the original signal structure. In the second step, if \( s \) originally induces the buyer to buy, then \( \tilde{s} = 1 \); otherwise, \( \tilde{s} = 0 \). Finally, define\(^{25}\)

\[
\tilde{p}(\theta, \tilde{s} = 1) = \frac{\int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\}} p(\theta, s)d\sigma_\theta(s)}{\sigma_\theta(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\})}; \\
\tilde{p}(\theta, \tilde{s} = 0) = \frac{\int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) < 0\}} p(\theta, s)d\sigma_\theta(s)}{\sigma_\theta(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) < 0\})}.
\]

We first show that \((\tilde{p}(\theta, s), (\tilde{S}_\theta, \tilde{\sigma}_\theta))\) is IC. For type \( \theta \), if he reports truthfully, his expected payoff is the same as before (LHS of (IC-1)) by our construction of \( \tilde{s} \) and \( \tilde{p} \). On the other hand, if he reports \( \hat{\theta} \neq \theta \), his expected payoff is weakly less than before (RHS of (IC-1)) because \((\{0, 1\}, \tilde{\sigma}_\theta)\) is a Blackwell garbling of \((S_\theta, \sigma_\theta)\). Therefore, (IC-1) is still satisfied by the new mechanism.

\(^{25}\) If the probability in the denominator is 0, define the price to be any finite number.
Moreover, in the new mechanism,

\[ \theta + \mathbb{E}(\omega|\tilde{s} = 1, \theta) - \hat{p}(\theta, \tilde{s} = 1) = \theta + \frac{\int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\}} [\mathbb{E}(\omega|s, \theta) - p(\theta, s)] d\sigma_{\theta}(s)}{\sigma_{\theta}(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\})} = \frac{\int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\}} [\theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s)] d\sigma_{\theta}(s)}{\sigma_{\theta}(\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\})} \geq 0, \text{ whenever } \sigma_{\theta}(\tilde{s} = 1) > 0; \]

\[ \theta + \mathbb{E}(\omega|\tilde{s} = 0, \theta) - \hat{p}(\theta, \tilde{s} = 0) < 0, \text{ whenever } \sigma_{\theta}(\tilde{s} = 0) > 0. \]

So the new mechanism is a recommendation mechanism with contractible signals.

Finally, the new mechanism generates the same profit because (using the definition of \( \hat{p} \) and \( \tilde{s}(s) \))

\[ \int_{\Theta} p(\theta, \tilde{s} = 1) \sigma_{\theta}(\tilde{s} = 1) dG(\theta) = \int_{\Theta} \left[ \int_{\{s: \theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s) \geq 0\}} (p(\theta, s) - c) d\sigma_{\theta}(s) \right] dG(\theta). \]

Proof of Theorem 2. Take any IC recommendation mechanism with contractible signals, denoted by \( \{p(\theta, 1), p(\theta, 0), q(\omega, \theta)\} \).

Consider another recommendation mechanism with contractible signals \( \{p(\theta, 1), \hat{p}(\theta, 0), q(\omega, \theta)\} \), such that

\[ \hat{p}(\theta, 0) = \tilde{\theta} + \tilde{\omega} + 1, \text{ for all } \theta \in [\tilde{\theta}, \hat{\theta}]. \]

That is, we set all \( \hat{p}(\theta, 0) \) to be strictly greater than \( \tilde{\theta} + \tilde{\omega} \), the highest possible willingness to pay.

This new mechanism is still IC because

- for each type \( \theta \), if he reports truthfully, his expected payoff from buying when \( s = 1 \) does not change, while his expected payoff from buying when \( s = 0 \) is always less than 0. So if reporting truthfully, he is willing to take the recommended action in the new mechanism;

- for each type \( \theta \), if he report \( \hat{\theta} \neq \theta \), his expected payoff (after maximizing over actions) when receiving \( \tilde{s} = 1 \) does not change, while his expected payoff when receiving \( \tilde{s} = 0 \) is weakly less because now it is 0 (as \( \hat{p}(\hat{\theta}, 0) \) is too high). So if he reports \( \hat{\theta} \), his expected payoff is weakly less than doing the same in the original mechanism.

This new mechanism generates the same expected profit as before, because for each type the price charged conditional on buying and the probability of buying \( (s = 1) \) do not change.

The above argument implies that it is without to focus on the set of recommendation mechanisms with contractible signals s.t. \( p(\theta, 0) = \tilde{\theta} + \tilde{\omega} + 1 \) for all \( \theta \). Moreover, for this set of mechanisms, even if type \( \theta \) reports \( \hat{\theta} \), he will never want to buy when \( \tilde{s} = 0 \) because \( p(\hat{\theta}, \tilde{s}) = \tilde{\theta} + \tilde{\omega} + 1 \). Therefore, its IC constraint can be written as:

\[ \int_{\omega} [\theta + \omega - p(\theta, 1)] q(\omega, \theta) dF(\omega) \geq \max \left\{ 0, \int_{\omega} [\theta + \omega - p(\hat{\theta}, 1)] q(\omega, \hat{\theta}) dF(\omega) \right\}, \forall \theta, \hat{\theta} \in [\tilde{\theta}, \hat{\theta}]. \]

(IC-Contractible)
That is, “reporting truthfully and taking the recommended action” should be better than “reporting another type and never buying” and “reporting another type and taking the recommended action”.

The seller’s program for this class of mechanisms is (15), which we copied below:

$$\max_{\{p(\theta, 1), q(\omega, \theta)\}} \mathbb{E}_{\theta} \left[ (p(\theta, 1) - c) \int_{\omega} q(\omega, \theta) dF(\omega) \right].$$

s.t. (IC-contractible)

Notice that this program is same as programs (5) and (6), except that

$$(IC) \Rightarrow (IC\text{-contractible}) \Rightarrow (IC').$$

We have shown in Theorem 1 that the solution to program (6) under (IC) solves program (5) under (IC), so we conclude that the same solution also solves program (15). □

**Lemma 4.** For any general mechanism with contractible signals that satisfies (IC-general-contractible) and (ex post IR), there exists an IC recommendation mechanism with contractible signals studied in Section 5.1 that generates the same profit.

*Proof.* Take any general mechanism with contractible signals

$$\{(S_\theta, \sigma_\theta), (X(\theta, s), T(\theta, s))\},$$

and suppose that it satisfies (IC-general-contractible) and (ex post IR). Since it is ex post IR, we know that whenever $X(\theta, s) = 0$, we have $T(\theta, s) = 0$. Define $p(\theta, s)$ by

$$p(\theta, s) = \begin{cases} \frac{T(\theta, s)}{X(\theta, s)}, & \text{if } X(\theta, s) > 0 \\ P, & \text{if } X(\theta, s) = 0 \end{cases},$$

where $P$ is a very large number greater than the buyer’s highest possible willingness to pay. Notice that $p(\theta, s)$ satisfies

$$T(\theta, s) = p(\theta, s)X(\theta, s), \text{ for all } \theta \in \Theta, s \in S_\theta.$$

Now define a direct mechanism with contractible signals $\{\tilde{p}(\theta, s), (\tilde{S}_\theta, \tilde{\sigma}_\theta)\}$ such that $\tilde{S}_\theta = \{0, 1\}$ for all $\theta$, and for each $\theta$, let $(\{0, 1\}, \tilde{\sigma}_\theta)$ be a garbling of $(S_\theta, \sigma_\theta)$ such that the signal realization $\tilde{s}$ from $(\{0, 1\}, \tilde{\sigma}_\theta)$ is determined by the signal realization $s$ from $(S_\theta, \sigma_\theta)$ in the following way:

$$\tilde{s}(s) = \begin{cases} 1, & \text{w.p. } X(\theta, s); \\ 0, & \text{w.p. } 1 - X(\theta, s). \end{cases}$$
In addition, let

$$\hat{p}(\theta, 1) = \frac{\int_{S_\theta} X(p, s)p(\theta, s)d\sigma_\theta(s)}{\int_{S_\theta} X(\theta, s)d\sigma_\theta(s)},$$

\(\hat{p}(\theta, 0) = P.\)

We will show that this direct mechanism with contractible signals is an IC recommendation mechanism with contractible signals, and generates the same profit as \(\{(S_\theta, \sigma_\theta), (X(\theta, s), T(\theta, s))\}\).

To prove that it is a recommendation mechanism, we need to show that for any type \(\theta\), if he reports truthfully, he is always willing to take the recommended action.

- If \(\theta\) is such that \(\int_{S_\theta} X(\theta, s)d\sigma_\theta(s) = 0\) (i.e. the buyer gets the object with probability 0 in the original mechanism), then by construction \(\tilde{s} = 0\) with probability 1; moreover, since \(p(\theta, 0) = P\) and \(P\) is very large, we know that the buyer never wants to buy in the new mechanism. In both mechanisms, this buyer gets an interim expected payoff of 0 (after knowing his type).

- If \(\theta\) is such that \(\int_{S_\theta} X(\theta, s)d\sigma_\theta(s) > 0\), then for all \(s \in S_\theta\) s.t. \(X(\theta, s) > 0\), ex post IR requires that

$$0 \leq X(\theta, s)(\theta + \mathbb{E}(\omega|s, \theta)) - T(\theta, s) = X(\theta, s)(\theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s)).$$

Then, when \(\tilde{s} = 1\), we have

$$0 \leq \frac{\int_{S_\theta} [\theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s)] X(\theta, s)d\sigma_\theta(s)}{\int_{S_\theta} X(\theta, s)d\sigma_\theta(s)}$$

$$= \theta + \mathbb{E}(\omega|\tilde{s} = 1, \theta) - \hat{p}(\theta, 1).$$

So the buyer is willing to buy when recommended so. When \(\tilde{s} = 0\), \(\hat{p}(\theta, 0) = P\), thus the buyer does not want to buy when recommended against so.

In this case, the buyer is always willing to take the recommended action, and his interim payoff is \(\int_{S_\theta} [\theta + \mathbb{E}(\omega|s, \theta) - p(\theta, s)] X(\theta, s)d\sigma_\theta(s)\) in both mechanisms.

To see that \(\{\hat{p}(\theta, s), (\tilde{S}_\theta, \tilde{\sigma}_\theta)\}\) is IC, let us analyze the buyer’s incentives to misreport in this recommendation mechanism. For any type \(\theta\), by the previous step, we know that if he reports truthfully, he will take the recommended action and get the same interim payoff as in the original mechanism. If he reports \(\hat{\theta}\), then if the realized signal is \(\tilde{s} = 0\), the buyer will not buy because \(\hat{p}(\hat{\theta}, 0) = P\) is very large. So after reporting \(\hat{\theta}\), the buyer should either always take the recommended action, or never buy. If the buyer always takes the recommended action, his

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26 If \(\int_{S_\theta} X(\theta, s)d\sigma_\theta(s) = 0\), set \(p(\theta, 1)\) to be any positive number.

27 Here, we use the notation \(\mathbb{E}(\omega|s, \theta)\) again. As in the main text, though we write \(\theta\) in the conditional part, it only means that signal \(s\) is realized from the signal structure \((S_\theta, \sigma_\theta)\) given report \(\theta\). The random variables \(\theta\) and \(\omega\) are independent.
interim payoff is the same as reporting \( \hat{\theta} \) in the original mechanism, because

\[
E_{s \sim \sigma_{\hat{\theta}}} \left[ X(\hat{\theta}, s)(\theta + E(\omega | s, \hat{\theta})) - T(\hat{\theta}, s) \right] = E_{s \sim \sigma_{\hat{\theta}}} \left[ X(\hat{\theta}, s)(\theta + E(\omega | s, \hat{\theta}) - p(\hat{\theta}, s)) \right]
\]

\[
= \int_{S_{\theta}} \left[ \theta + E(\omega | s, \hat{\theta}) - p(\hat{\theta}, s) \right] X(\hat{\theta}, s) d\sigma_{\hat{\theta}}(s)
\]

\[
= \left[ \theta + \frac{\int_{S_{\theta}} E(\omega | s, \hat{\theta}) X(\hat{\theta}, s) d\sigma_{\hat{\theta}}(s)}{\tilde{\sigma}_{\hat{\theta}}(\tilde{s} = 1)} - \tilde{p}(\hat{\theta}, 1) \right] \tilde{\sigma}_{\hat{\theta}}(\tilde{s} = 1)
\]

where the LHS is the interim payoff from reporting \( \hat{\theta} \) in the original mechanism, and the last term is the interim payoff from reporting \( \hat{\theta} \) and following recommendations in the new mechanism, as desired. If the buyer never buys, his payoff is 0. Since type \( \theta \) does not want to report \( \hat{\theta} \) in the original mechanism, neither does he in the new one.

Finally, we show that \( \{\tilde{p}(\theta, s), (\tilde{S}_{\theta}, \tilde{\sigma}_{\theta})\} \) generates the same profit as the original general mechanism with contractible signals. The profit generated by the original mechanism satisfies

\[
E_{\theta} \left[ \int_{S_{\theta}} (T(\theta, s) - c) d\sigma_{\theta}(s) \right] = E_{\theta} \left[ \int_{S_{\theta}} (X(\theta, s) p(\theta, s) - c) d\sigma_{\theta}(s) \right]
\]

\[
= E_{\theta} [\tilde{p}(\theta, 1) - c] \tilde{\sigma}_{\theta}(\tilde{s} = 1),
\]

where the first line follows from \( T(\theta, s) = p(\theta, s) X(\theta, s) \) as in (22), and the last line follows from the definition of \( \tilde{p}(\theta, 1) \) in (23). Note that the last term is the profit generated by \( \{\tilde{p}(\theta, s), (\tilde{S}_{\theta}, \tilde{\sigma}_{\theta})\} \), as desired. \( \square \)

**Lemma 5.** An optimal ex post IR general mechanism with contractible signals generates a weakly higher profit than any (optimal) ex post IR general mechanism.

**Proof.** This follows directly from the fact that (IC-general) implies (IC-general-contractible). That is, any IC general mechanism is also an IC general mechanism with contractible signals.\(^{28}\) Since ex post IR and the seller’s objective have exactly the same forms regardless of the contractibility of signals, we have the claimed result. \( \square \)

**Proof of Theorem 3.** Let us first transform the mechanism described in Theorem 1 into a general mechanism that satisfies (ex post IR) and (IC-general). Specifically, define a general mechanism \( \{(S_{\theta}^*, \sigma_{\theta}^*), (X^*(\theta, s), T^*(\theta, s))\} \) as follows. Let

\[
S_{\theta}^* = \{0, 1\}, \text{ for all } \theta \in \Theta;
\]

\[
\sigma_{\theta}^*(s = 1; \omega) = q^*(\omega, \theta);
\]

\[
X^*(\theta, s) = 1_{\{\theta + E(\omega | s, \theta) - p^*(\theta) \geq 0\}};
\]

\[
T^*(\theta, s) = X^*(\theta, s) p^*(\theta).
\]

\(^{28}\) More precisely, if \( \{(S_{\theta}, \sigma_{\theta}), (X(\theta, s), T(\theta, s))\} \) is IC (i.e. satisfies (IC-general)) when understood as a general mechanism, then \( \{(S_{\theta}, \sigma_{\theta}), (X(\theta, s), T(\theta, s))\} \) is IC (i.e. satisfies (IC-general-contractible)) when understood as a general mechanism with contractible signals.

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Incentive compatibility of \( \{p^*, q^*\} \) implies that \( \{(S^*_\theta, \sigma^*_\theta), (X^*(\theta, s), T^*(\theta, s))\} \) defined above satisfies (IC-general).

By construction of \( X \) and \( T \), it is ex post IR; and it generates the same profit as \( \{p^*, q^*\} \).

By Lemma 4 and Theorem 2, we know that \( \{p^*, q^*\} \) and the above mechanism weakly dominate all ex post IR general mechanisms with contractible signals. Lemma 5 then implies that the above mechanism is an optimal ex post IR general mechanism.

\[ \square \]

### A.3 Proofs of Results in Section 5.3

**Proof of Theorem 4.** By Lemma 1,\(^{29}\) it is without loss to focus on recommendation mechanisms of the form \( \{p(\theta), q(\omega, \theta)\}_{\theta \in \Theta} \).

Let us first define \( \bar{M} \) and \( \tilde{M} \). Define the following function

\[
v(\theta, \omega) \equiv u(\theta, \omega) - \frac{1 - G(\theta)}{g(\theta)} u_\theta(\theta, \omega) - c. \tag{24}\]

Notice that

\[

v_\theta = u_\theta - \left(\frac{1 - G}{g}\right)' u_\theta - \frac{1 - G}{g} u_{\theta\theta} > 0, \tag{25}

v_\omega = u_\omega - \frac{1 - G}{g} u_{\theta \omega}, \tag{26}

\]

where the inequality follows from \( u_\theta > 0, \left(\frac{1 - G}{g}\right)' < 0 \) and \( u_{\theta\theta} \leq 0 \). Since \( v_\theta > 0 \), let \( \theta(\omega) \) be such

\[

\theta(\omega) \equiv \sup\{\theta : v(\theta, \omega) \leq 0\}. \tag{27}

\]

Note that \( \theta(\omega) \) is continuous in \( \omega \). Let \( \hat{\theta} \equiv \min_{\omega \in [\omega_1, \omega]} \theta(\omega) \).

Define

\[

\bar{M} = \min_{\theta \in [\hat{\theta}, \bar{\theta}]} u_\omega(\theta, \bar{\omega}) \frac{g(\theta)}{1 - G(\theta)}; \tag{27}

\]

\[

\tilde{M} = \max_{(\theta, \omega) \in \Theta \times \Omega} u_\omega(\theta, \bar{\omega}) \left[ \frac{\partial}{\partial \theta} \ln \left( \frac{u_\theta(\theta, \omega)\left(\frac{1 - G(\theta)}{g(\theta)}\right)}{\theta(\omega)} \right) \right]. \tag{28}

\]

Notice that \( \bar{M} < 0 < \tilde{M} \), because for all \( (\theta, \omega) \in \Theta \times \Omega \), we have that \( u_\omega > 0, u_\theta > 0, u_{\theta\theta} < 0 \) and \( \frac{\partial}{\partial \theta} \left(\frac{1 - G}{g}\right) < 0 \), and that all functions in above expressions are continuous.\(^{30}\)

To prove Theorem 4, let us assume

\[

\bar{M} < u_{\theta\omega}(\theta, \omega) < \tilde{M}, \text{ for all } (\theta, \omega) \in \Theta \times \Omega. \tag{29}

\]

From (26), we can see that if \( u_{\theta\omega} < \bar{M} \) for all \( \theta \) and \( \omega \), then \( v_\omega > 0 \) whenever \( v \geq 0 \). This upper bound is

\[\footnotesize\text{29} \] The proof of Lemma 1 can be easily extended to the case without additive separability.

\[\footnotesize\text{30} \] To be more precise, here we additionally assume that \( g \) is continuously differentiable.

Electronic copy available at: https://ssrn.com/abstract=3263898
satisfied, for example, when \( u(\theta, \omega) = \theta \omega \).

Define

\[
U(\theta, \hat{\theta}) \equiv \int_{\omega}^{\hat{\omega}} [u(\theta, \omega) - p(\hat{\theta})]q(\omega, \hat{\theta})dF(\omega) = C(\hat{\theta}) + D(\theta, \hat{\theta}),
\]

\[
V(\theta) \equiv U(\theta, \theta),
\]

where

\[
C(\hat{\theta}) = -p(\hat{\theta}) \int_{\omega}^{\hat{\omega}} q(\omega, \hat{\theta})dF(\omega),
\]

\[
D(\theta, \hat{\theta}) = \int_{\omega}^{\hat{\omega}} u(\theta, \omega)q(\omega, \hat{\theta})dF(\omega).
\]

Analogous to (IC), the buyer’s incentive compatibility constraint can be written as

\[
V(\theta) \geq \max_{\hat{\theta} \in \Theta} \left\{ 0, \mathbb{E}_\omega[u(\theta, \omega)] - p(\hat{\theta}), U(\theta, \hat{\theta}) \right\}, \text{ for all } \theta \in \Theta. \tag{IC-nonadditive}
\]

The seller’s program is

\[
\max_{\{p(\theta), q(\omega, \theta)\}} \mathbb{E}_\theta \left[ (p(\theta) - c) \int_{\omega}^{\hat{\omega}} q(\omega, \theta)dF(\omega) \right], \tag{31}
\]

s.t. (IC-nonadditive)

**Seller’s Relaxed Program**

As in the additively separable case, we first consider the following relaxed constraint

\[
V(\theta) \geq U(\theta, \hat{\theta}), \text{ for all } \theta, \hat{\theta} \in \Theta, \tag{IC-nonadditive’}
\]

and the seller’s relaxed program

\[
\max_{\{p(\theta), q(\omega, \theta)\}} \mathbb{E}_\theta \left[ (p(\theta) - c) \int_{\omega}^{\hat{\omega}} q(\omega, \theta)dF(\omega) \right], \tag{32}
\]

s.t. (IC-nonadditive’)

**Claim 1.** A recommendation mechanism satisfies (IC-nonadditive’) only if

\[
V(\theta) = V + \int_{\hat{\theta}}^{\theta} D_1(s, s)ds, \quad \text{where } D_1(\theta, \hat{\theta}) = \frac{\partial D(\theta, \hat{\theta})}{\partial \theta}. \tag{33}
\]

Moreover, if a recommendation mechanism satisfies (33) and that

\[D_1(\theta, \hat{\theta})\text{ is nondecreasing in } \hat{\theta}, \text{ for all } \theta, \hat{\theta} \in \Theta,\]
then it satisfies (IC-nonadditive’).

Proof. Take any recommendation mechanism that satisfies (IC-nonadditive’). Recall that

\[ D(\theta, \hat{\theta}) = \int_{\omega} u(\theta, \omega) q(\omega, \hat{\theta}) dF(\omega). \]

Since \( u \) is continuously differentiable and \( u_\theta(\theta, \omega) q(\omega, \hat{\theta}) f(\theta) \leq \max_{(\theta, \omega) \in \Theta \times \Omega} [u_\theta(\theta, \omega) f(\theta)] \), by the Dominated Convergence Theorem,

\[ D_1(\theta, \hat{\theta}) = \int_{\omega} u_\theta(\theta, \omega) q(\omega, \hat{\theta}) dF(\omega). \]

Moreover, \(|D_1(\theta, \hat{\theta})| \leq \int_{\omega} u_\theta(\theta, \omega) dF(\omega) \leq \max_{(\theta, \omega) \in \Theta \times \Omega} u_\theta(\theta, \omega)\), so by Theorem 2 in Milgrom and Segal (2002),

\[ V(\theta) = V + \int_{\theta} D_1(s, s) ds. \]

Now suppose that a recommendation mechanism satisfies (33) and that \( D_1(\theta, \hat{\theta}) \) is nondecreasing in \( \hat{\theta} \), for all \( \theta, \hat{\theta} \in \Theta \). We want to show that (IC-nonadditive’) is satisfied. Take any \( \theta, \hat{\theta} \in \Theta \). We have

\[ V(\theta) - U(\theta, \hat{\theta}) = U(\theta, \theta) - U(\theta, \hat{\theta}) \]
\[ = V(\theta) - V(\hat{\theta}) + D(\hat{\theta}, \hat{\theta}) - D(\theta, \hat{\theta}) \]
\[ = \int_{\theta}^{\hat{\theta}} D_1(s, s) ds - \int_{\theta}^{\hat{\theta}} D_1(s, \hat{\theta}) ds \]
\[ = \int_{\theta}^{\hat{\theta}} [D_1(s, s) - D_1(s, \hat{\theta})] ds \]
\[ \geq 0, \]

where the second line follows from equations (29) and (30), the third line follows from condition (33), and the last line follows from \( D_1 \) being nondecreasing in its second variable. \( \square \)

By Claim 1, we have

\[ V + \int_{\theta}^{\hat{\theta}} D_1(s, s) ds = V(\theta) = C(\theta) + D(\theta, \theta) \]

Since \( C(\theta) = -p(\theta) \int_{\omega} q(\omega, \theta) dF(\omega) \), we have

\[ p(\theta) \int_{\omega} q(\omega, \theta) dF(\omega) = -V + D(\theta, \theta) - \int_{\theta}^{\hat{\theta}} D_1(s, s) ds. \] (34)
So the seller’s relaxed program can be written as

$$\max_{\{q(\omega, \theta) : \mathcal{V}\}} - V + \int_\theta^\bar{\theta} \left[ D(\theta, \theta) - \int_\theta^\theta D_1(s, s)ds - c \int_\omega^\bar{\omega} q(\omega, \theta)dF(\omega) \right] dG(\theta).$$

(35)

s.t. $D_1(\theta, \bar{\theta})$ is nondecreasing in $\bar{\theta}$, and $V \geq 0$

Note that

$$\int_\theta^\bar{\theta} \left[ D(\theta, \theta) - \int_\theta^\theta D_1(s, s)ds - c \int_\omega^\bar{\omega} q(\omega, \theta)dF(\omega) \right] dG(\theta)$$

$$= \int_\theta^\bar{\theta} \int_\omega^\bar{\omega} \left( u(\theta, \omega) - c - \frac{1 - G(\theta)}{g(\theta)} u_{\theta, \theta}(\theta, \omega) \right) q(\omega, \theta)g(\theta)dF(\omega)d\theta$$

$$= \int_\theta^\bar{\theta} \int_\omega^\bar{\omega} v(\theta, \omega)q(\omega, \theta)g(\theta)dF(\omega)d\theta,$$

(36)

where the second line is obtained by integration by parts and substituting the expressions of $D$ and $D_1$ into the equation, and the last line uses the definition of $v$ in (24).

From (25), (26) and the assumption that $u_{\theta, \omega} < M$, we know that

$$v_\theta, v_\omega > 0, \text{ whenever } v \geq 0.$$

(37)

So for any given $\theta$, $v(\theta, \cdot)$ has at most one zero point.

Let $k(\theta)$ be s.t.

$$v(\theta, k(\theta)) = 0.$$

whenever well-defined. In addition, $k(\theta) \equiv \bar{\omega}$ ($k(\theta) \equiv \underline{\omega}$) if $v(\theta, \omega)$ is negative (positive) for all $\omega$.

Condition (37) implies that

$$\frac{dk(\theta)}{d\theta} = \frac{v_\theta}{v_\omega} < 0,$$

(38)

for all $\theta$ s.t. $v(\theta, \cdot)$ admits a zero point in $\Omega$.

The derivation in (36) suggests the following candidate solution:

$$q^*(\omega, \theta) = \begin{cases} 1, & \text{if } \omega \geq k(\theta) \\ 0, & \text{if } \omega < k(\theta) \end{cases}$$

(39)

Under such a candidate solution, $D^*(\theta, \bar{\theta}) = \int_{\tilde{\omega}(\bar{\theta})}^{\bar{\omega}} u(\theta, \omega)dF(\omega)$, so that

$$\frac{\partial^2 D^*}{\partial \theta \partial \omega} = -u_\theta(\theta, \omega)\frac{dk(\theta)}{d\theta} > 0,$$

which verifies that $D^*_1(\theta, \bar{\theta})$ is nondecreasing in $\bar{\theta}$. Hence, $q^*$ defined in (39) solves program (35).

With an abuse of notation, define $\theta_1 = \inf_\Theta \{ \theta : k(\theta) < \underline{\omega} \}$ and $\theta_2 = \sup_\Theta \{ \theta : k(\theta) > \bar{\omega} \}$. Since we assumed $u(\bar{\theta}, \bar{\omega}) > c$ and $u(\underline{\theta}, \underline{\omega}) \leq c$, it is easy to check that $\bar{\theta} \leq \theta_1 < \theta_2 \leq \bar{\theta}$.

As in the additively separable case, all buyer types between $\theta_1$ and $\theta_2$ buys with probability strictly between 0 and 1.

\[\text{31} \text{ That } u(\bar{\theta}, \bar{\omega}) > c \text{ and } u(\underline{\theta}, \underline{\omega}) \leq c \text{ implies that } k(\bar{\theta}) < \bar{\omega} \text{ and } k(\underline{\theta}) > \underline{\omega}, \text{ so that } \theta_1 \text{ (} \theta_2 \text{) is well-defined.}\]
By equation (34), the implied pricing schedule is:

\[
p^*(\theta) = \frac{D^*(\theta, \theta) - \int_{\theta}^{\theta_1} D_1^*(s, s) ds}{1 - F(k(\theta))}
= E[u(\theta, \omega)|\omega \geq k(\theta)] - \frac{\int_{\theta}^{\theta_1} D_1^*(s, s) ds}{1 - F(k(\theta))}, \forall \theta \in [\theta_1, \bar{\theta}]
\]

(40)

When \(\theta_1 > \theta\), we can without loss set \(p^*(\theta) = u(\theta_1, \bar{\omega})\).

**Seller’s Original Program**

We now argue that \(\{p^*, q^*\}\) defined in (39) and (40) solves the seller’s original program (31).

**Claim 2.** \(p^*(\theta)\) is decreasing in \(\theta\).

**Proof.** By its definition in (40), \(p^*\) is constant on \([\theta, \theta_1]\) and \([\theta_2, \bar{\theta}]\), and is continuous at \(\theta_2\). For the possible case where \(\theta_1 > \theta\), let us first show that \(p^*\) is continuous at \(\theta_1\). Note that

\[
\lim_{\theta \to \theta_1} p^*(\theta) = \lim_{\theta \to \theta_1} \left[ \frac{D^*(\theta, \theta) - \int_{\theta}^{\theta_1} D_1^*(s, s) ds}{1 - F(k(\theta))} \right]
= \frac{D_1^*(\theta_1, \theta_1) + D_2^*(\theta_1, \theta_1) - D_1^*(\theta_1, \theta_1)}{-f(k(\theta_1))k'(\theta_1)}
= u(\theta_1, \bar{\omega}),
\]

where the second line follows from L’Hopital’s rule, and the third line follows from \(D_2^*(\theta, \theta) = -u(\theta, k(\theta))f(k(\theta))k'(\theta)\).

Now we show that \(p^*\) is strictly decreasing on \((\theta_1, \theta_2)\). Taking derivative with respect to \(\theta\), we have

\[
\frac{dp^*(\theta)}{d\theta} = \frac{d}{d\theta} \left[ E[u(\theta, \omega)|\omega \geq k(\theta)] - \frac{\int_{\theta}^{\theta_1} D_1^*(s, s) ds}{1 - F(k(\theta))} \right]
= \int_{k(\theta)}^{\omega} u_\theta(\theta, \omega) dF(\omega) \frac{f(k(\theta)) \int_{k(\theta)}^{\omega} (1 - F(\omega))u_\omega(\theta, \omega) d\omega}{[1 - F(k(\theta))]^2} k'(\theta)
- \left[ \frac{D_1^*(\theta, \theta)}{1 - F(k(\theta))} + \frac{f(k(\theta)) \int_{k(\theta)}^{\theta} D_1^*(s, s) ds}{[1 - F(k(\theta))]^2} k'(\theta) \right]
= -\frac{f(k(\theta))k'(\theta)}{[1 - F(k(\theta))]^2} \left[ \int_{\theta_1}^{\theta} D_1^*(s, s) ds - \int_{k(\theta)}^{\omega} (1 - F(\omega))u_\omega(\theta, \omega) d\omega \right],
\]

where the second line follows from the observation (analogous to Observation 1) that

\[
\frac{d}{dy} E[u(\theta, x)|x \geq y] = \frac{f(y) \int_{y}^{x} (1 - F(x))u_x(\theta, x) dx}{(1 - F(y))^2},
\]

and the last line follows from \(D_1^*(\theta, \theta) = \int_{k(\theta)}^{\omega} u_\theta(\theta, \omega) dF(\omega)\).
To show \( dp^*/d\theta < 0 \), because \( k'(\theta) < 0 \), we are done if

\[
\frac{d}{d\theta} \left[ \int_{\theta_1}^{\theta} D_1^*(s, s) ds - \int_{k(\theta)}^{\bar{\omega}} (1 - F(\omega)) u_\omega(\theta, \omega) d\omega \right] < 0
\]

for all \( \theta \in (\theta_1, \theta_2) \). Since \( k(\theta_1) \leq \bar{\omega} \), these integrals are weakly less than 0 at \( \theta = \theta_1 \). Moreover, for all \( \theta \in [\theta_1, \theta_2] \),

\[
\frac{d}{d\theta} \left[ \int_{\theta_1}^{\theta} D_1^*(s, s) ds - \int_{k(\theta)}^{\bar{\omega}} (1 - F(\omega)) u_\omega(\theta, \omega) d\omega \right] = D_1^*(\theta, \theta) - \left( \int_{k(\theta)}^{\bar{\omega}} (1 - F(\omega)) u_{\theta\omega}(\theta, \omega) d\omega - [1 - F(k(\theta))] u_\omega(\theta, k(\theta)) k'(\theta) \right)
\]

\[
= [1 - F(k(\theta))] \left[ u_\theta(\theta, k(\theta)) + k'(\theta) u_\omega(\theta, k(\theta)) \right]
\]

\[
= [1 - F(k(\theta))] \left[ -u_\theta \frac{1-G}{g} u_{\theta\omega} + u_\omega \left( \frac{1-G}{g} \right) u_\theta + \frac{1-G}{g} u_{\theta\theta} \right]
\]

\[
< 0,
\]

where the second equality follows from doing integration by parts in \( D_1^* \), the third and fourth equalities follow from conditions (37) and (38), and the strict inequality at the end follows from \( u_{\theta\omega} > M \). Therefore, we have

\[
\int_{\theta_1}^{\theta} D_1^*(s, s) ds - \int_{k(\theta)}^{\bar{\omega}} (1 - F(\omega)) u_\omega(\theta, \omega) d\omega < 0, \text{ for all } \theta \in (\theta_1, \bar{\theta}),
\]

as desired. \( \square \)

Now we prove that \( \{p^*, q^*\} \) satisfies the original IC constraint (IC-nonadditive). First, since \( V = 0 \), Lemma 1 implies that \( V(\theta) \geq 0 \) for all \( \theta \in [\theta, \bar{\theta}] \).

Second, we show that, for all \( \theta \in [\theta, \bar{\theta}] \), \( \mathbb{E}[u(\theta, \omega)|\omega < k(\theta)] - p^*(\theta) \leq 0 \); that is, when recommended not to buy, each type finds it optimal to follow the recommendation. Note that, for \( \theta \in [\theta_1, \bar{\theta}] \),

\[
\mathbb{E}[u(\theta, \omega)|\omega < k(\theta)] - p^*(\theta) \leq u(\theta, k(\theta)) - p^*(\theta)
\]

\[
= u(\theta, k(\theta)) - \left[ \mathbb{E}[u(\theta, \omega)|\omega \geq k(\theta)] - \frac{\int_{\theta_1}^{\theta} D_1^*(s, s) ds}{1 - F(k(\theta))} \right]
\]

\[
= \frac{1}{1 - F(k(\theta))} \left( \int_{\theta_1}^{\theta} D_1^*(s, s) ds - \int_{k(\theta)}^{\bar{\omega}} (1 - F(\omega)) u_\omega(\theta, \omega) d\omega \right)
\]

\[
\leq 0
\]

(41)

where the last inequality follows from our analysis of the terms in the parenthesis in the proof of Claim 2. For \( \theta \in [\theta, \theta_1) \) if any, they do not want to buy when recommended against so because this is true for type \( \theta_1 \).

To prove that \( \{p^*, q^*\} \) satisfies (IC-nonadditive), it remains to verify that \( V(\theta) \geq \max_\theta \mathbb{E}_\omega[u(\theta, \omega)] - p(\theta) \).
By Claim 2, \( \min_{\hat{\theta}} p(\hat{\theta}) = p(\bar{\theta}) \). So for any \( \theta \in [\hat{\theta}, \bar{\theta}] \), we have
\[
\max_{\hat{\theta}} E_{\omega}[u(\theta, \omega)] - p(\hat{\theta}) = E_{\omega}[u(\theta, \omega)] - p(\hat{\theta})
\]
\[
= [1 - F(k(\hat{\theta}))] [E_{\omega}[u(\theta, \omega)|\omega \geq k(\hat{\theta})] - p(\hat{\theta})] + F(k(\hat{\theta})) [E_{\omega}[u(\theta, \omega)|\omega < k(\hat{\theta})] - p(\hat{\theta})]
\]
\[
\leq [1 - F(k(\hat{\theta}))] [E_{\omega}[u(\theta, \omega)|\omega \geq k(\hat{\theta})] - p(\hat{\theta})] + F(k(\hat{\theta})) [E_{\omega}[u(\theta, \omega)|\omega < k(\hat{\theta})] - p(\hat{\theta})]
\]
\[
\leq [1 - F(k(\hat{\theta}))] [E_{\omega}[u(\theta, \omega)|\omega \geq k(\hat{\theta})] - p(\hat{\theta})]
\]
\[
= U(\theta, \bar{\theta})
\]
where the first inequality follows from \( \theta \leq \bar{\theta} \), and the second inequality follows from (41). Therefore, \((p^*, q^*)\) satisfies (IC-nonadditive).

\[\square\]

A.4 Proof of Results in Section 6

Similar to our previous derivation, given a mechanism \( \{p(\theta), \mu(\theta), q(\epsilon, \theta)\} \), let us define
\[
U(\theta, \hat{\theta}) \equiv \int_{\epsilon}^\bar{\epsilon} \left[ \mu(\hat{\theta}) + \epsilon \right] dF(\epsilon)
\]
\[
= \max_{\hat{\theta}} E_{\omega}[u(\theta, \omega)] - p(\hat{\theta})
\]
\[
= 1 - F(k(\hat{\theta})) [E_{\omega}[u(\theta, \omega)|\omega \geq k(\hat{\theta})] - p(\hat{\theta})]
\]
\[
\leq 1 - F(k(\hat{\theta})) [E_{\omega}[u(\theta, \omega)|\omega \geq k(\hat{\theta})] - p(\hat{\theta})]
\]
\[
\leq U(\theta, \bar{\theta})
\]
where \( C(\hat{\theta}) \equiv -p(\hat{\theta}) \int_{\epsilon}^\bar{\epsilon} q(\epsilon, \hat{\theta}) dF(\epsilon) \), and
\[
D(\theta, \hat{\theta}) \equiv \int_{\epsilon}^\bar{\epsilon} [\mu(\hat{\theta}) + \epsilon] q(\epsilon, \hat{\theta}) dF(\epsilon)
\]
Taking double deviations into account, analogous to (IC-nonadditive), the IC constraint is
\[
V(\theta) \geq \max_{\hat{\theta}} \left\{ 0, \theta \mu(\hat{\theta}) - p(\hat{\theta}), U(\theta, \hat{\theta}) \right\}.
\]

The seller’s program is
\[
\max_{\{p(\theta), \mu(\theta), q(\epsilon, \theta)\}} E_{\theta} \left[ [p(\theta) - c(\mu(\theta))] \int_{\epsilon}^\bar{\epsilon} q(\epsilon, \theta) dF(\epsilon) \right]
\]
\[
\quad \text{s.t. (IC-quality)}
\]

Again, we will first solve a relaxed program, ignoring double deviations, and then verify that the candidate solution satisfies the original IC constraint (IC-quality).
Seller’s Relaxed Program

Consider the following relaxed constraint

\[ V(\theta) \geq U(\theta, \hat{\theta}), \text{ for all } \theta, \hat{\theta} \in \Theta, \]

(IC-quality’)

and the seller’s relaxed program

\[
\max_{\{p(\theta), \mu(\theta), q(\epsilon, \theta)\}} \mathbb{E}_{\theta} \left[ [p(\theta) - c(\mu(\theta))] \int_{\epsilon}^{\hat{\epsilon}} q(\epsilon, \theta) dF(\epsilon) \right].
\]

\[ \text{s.t. (IC-quality’)} \]

(43)

Claim 2 is applicable to characterizing (IC-quality’), so that program (43) can be reduced to

\[
\max_{\{\mu(\theta), q(\epsilon, \theta)\}} \int_{0}^{\theta} \int_{\xi}^{\epsilon} q(\omega, \theta) \left[ \left( \theta - \frac{1 - G(\theta)}{g(\theta)} \right) (\mu(\theta) + \epsilon) - c(\mu(\theta)) \right] g(\theta) dF(\omega)d\theta;
\]

\[ \text{s.t. } D_{1}(\theta, \hat{\theta}) \text{ is nondecreasing in } \hat{\theta} \]

with the pricing schedule satisfying (17).

Let \( \theta_{1} \) be such that

\[
\max_{\mu} \left( \theta_{1} - \frac{1 - G(\theta_{1})}{g(\theta_{1})} \right) (\mu + \epsilon) - c(\mu) = 0.
\]

That is, \( \theta_{1} \) is the type above which the virtual surplus can be positive for some realization of \( \epsilon \) while below which the virtual surplus is always negative no matter what \( \mu \) and \( \epsilon \) are. Note that \( \theta_{1} > 0 \).

Pointwise maximization leads to the solution described in Theorem 5, in which the quality choice satisfies

\[
\mu^{*}(\theta) = \max \left\{ \mu, c^{-1} \left( \theta - \frac{1 - G(\theta)}{g(\theta)} \right) \right\},
\]

(44)

the threshold in the recommendation rule satisfies

\[
m(\theta) = \frac{c(\mu^{*}(\theta))}{\theta - \frac{1 - G(\theta)}{g(\theta)}} - \mu^{*}(\theta),
\]

(45)

and the pricing schedule satisfies (20) copied below

\[
p^{*}(\theta) = \mathbb{E} [\theta(\mu(\theta) + \epsilon) | \epsilon \geq m(\theta)] - \frac{\int_{0}^{\theta} D_{1}^{*}(s, s) ds}{1 - F(m(\theta))}.
\]

Let us verify that the implied \( D_{1}^{*}(\theta, \hat{\theta}) = \int_{m(\theta)}^{\epsilon}(\mu^{*}(\hat{\theta}) + \epsilon)dF(\epsilon) \) is nondecreasing in \( \hat{\theta} \). Because of Assumption 2 and strict convexity of \( c, \mu^{*}(\theta) \geq 0, \) strictly so whenever \( \mu^{*}(\theta) > \mu \). With respect to the monotonicity of
Then, $D_{12}^*(\theta, \bar{\theta}) = \mu''(\bar{\theta}) \left[ 1 - F(m(\bar{\theta})) \right] - m'(\bar{\theta}) \left[ \mu^*(\bar{\theta}) + m(\bar{\theta}) \right] > 0.$

because $\mu'' \geq 0$, $m' < 0$, and by condition (45) $\mu^*(\bar{\theta}) + m(\bar{\theta}) = \frac{c(\mu^*(\bar{\theta}))}{\theta - \frac{1 - G(\bar{\theta})}{g(\bar{\theta})}} > 0.$ Therefore, the mechanism described in Theorem 5 is a solution to the relaxed program (43).

**Seller’s Original Program**

To prove that the same mechanism solves the original program, we need to verify that it satisfies the original IC constraint (IC-quality). Note first that under the candidate mechanism, type $\theta_1$ is never recommended to buy, and thus $U(\theta, \theta_1) = 0$ for all $\theta$. This means that, as before, we only need to check double deviations where the buyer first misreports his type, and then always buys regardless of the recommendation.

We first prove Proposition 5 which decomposes price variation into two parts. Then, we show that it is never optimal for any buyer to under-reports his type. Finally, we establish that after over-reporting his type, the buyer should at least follow the recommendation of “not buying”, so that double deviations are never optimal.

**Proof of Proposition 5.** From equation (20), for $\theta > \theta_1$, we have

$$p^*(\theta) = \frac{\int_m^\theta \left[ \theta (\mu^*(\theta) + \epsilon) \right] dF(\epsilon)}{1 - F(m(\theta))} - \frac{\int_{\theta_1}^\theta D_1^*(s, s) ds}{1 - F(m(\theta))}.$$

Then,

$$\frac{dp^*(\theta)}{d\theta} = \frac{\int_m^\theta [\mu^*(\theta) + \epsilon + \theta \mu''(\theta)] dF(\epsilon)}{1 - F(m(\theta))} + \frac{f(m(\theta)) \int_m^\theta [1 - F(\epsilon)] d\epsilon}{\left[ 1 - F(m(\theta)) \right]^2} m'(\theta)$$

$$- \left( \frac{D_1^*(\theta, \theta)}{1 - F(m(\theta))} + \frac{f(m(\theta)) \int_{\theta_1}^{\theta} D_1^*(s, s) ds}{\left[ 1 - F(m(\theta)) \right]^2} m'(\theta) \right),$$

$$= \theta \mu''(\theta) + I(\theta),$$

32 We only need to check monotonicity of $D_1^*$ for $\theta \geq \bar{\theta}$ (in which case $\bar{\theta} - \frac{1 - G(\bar{\theta})}{g(\bar{\theta})} > 0$), because $D_1(\theta, \bar{\theta}) \equiv 0$ for $\bar{\theta} < \theta$. 

Theorem 5
where

$$I(\theta) \equiv \frac{f(m(\theta))m'(\theta)}{[1 - F(m(\theta))]^2} \left[ \int_{m(\theta)}^{\bar{\theta}} \theta \left[1 - F(\epsilon)\right] d\epsilon - \int_{\theta_1}^{\theta} D^*_1(s, s) ds \right].$$  \hspace{1cm} (46)

Since $m'(\theta) < 0$, in order to establish that $I(\theta) < 0$ for all $\theta > \theta_1$, it is sufficient to show that

$$\int_{m(\theta)}^{\bar{\theta}} \theta \left[1 - F(\epsilon)\right] d\epsilon - \int_{\theta_1}^{\theta} D^*_1(s, s) ds > 0, \forall \theta > \theta_1.$$  \hspace{1cm} (47)

Note that the LHS of (47) is equal to 0 at $\theta = \theta_1$ because $m(\theta_1) = \bar{\epsilon}$. Moreover, for $\theta > \theta_1$,

$$\frac{d}{d\theta} \left[ \int_{m(\theta)}^{\bar{\theta}} \theta \left[1 - F(\epsilon)\right] d\epsilon - \int_{\theta_1}^{\theta} D^*_1(s, s) ds \right] = \int_{m(\theta)}^{\bar{\theta}} \theta \left[1 - F(\epsilon)\right] d\epsilon - \theta \left[1 - F(m(\theta))\right] m'(\theta) - D^*_1(\theta, \theta)$$

$$= \int_{m(\theta)}^{\bar{\theta}} \theta \left[1 - F(\epsilon)\right] d\epsilon - \theta \left[1 - F(m(\theta))\right] m'(\theta) - \int_{m(\theta)}^{\bar{\theta}} [\mu^* + \epsilon] dF(\epsilon)$$

$$= -\left[1 - F(m(\theta))\right] \left[\mu^* + m(\theta)\right] - \theta \left[1 - F(m(\theta))\right] m'(\theta)$$

$$= -\left[1 - F(m(\theta))\right] \left[\mu^* + m(\theta) + \theta m'(\theta)\right],$$

where the third line comes from integration by parts on the last term of the second line. (44) and (45) imply that

$$\mu^* + m(\theta) + \theta m'(\theta) = \frac{c(\mu^*)}{\theta - \frac{1 - G(\theta)}{g(\theta)}} - \frac{c(\mu^*)}{\theta - \frac{1 - G(\theta)}{g(\theta)}} \left[1 - \left(\frac{1 - G(\theta)}{g(\theta)}\right)\right]$$

$$= \frac{c(\mu^*)}{\left(\theta - \frac{1 - G(\theta)}{g(\theta)}\right)^2} \left[1 - \frac{1 - G(\theta)}{g(\theta)} + \theta \left(\frac{1 - G(\theta)}{g(\theta)}\right)^2\right]$$

$$< 0,$$

where the inequality follows from $\left(\frac{1 - G(\theta)}{g(\theta)}\right) < 0$. The above derivation implies that (47) holds, as desired. \qed

**Claim 3.** For any type of buyer $\theta > \theta_1$, it is never optimal to report $\hat{\theta} \leq \theta$.

Proof. (IC-quality') implies that it is never profitable to misreport and then always follow the recommendation. Since $U(\theta, \theta_1) = 0$ and $V(\theta) \geq 0$, it is never profitable to misreport and then never buy. Suppose type $\theta$ reports $\hat{\theta}$ and then always buys. His payoff is $\theta \mu^*(\hat{\theta}) - p^*(\hat{\theta})$. By Proposition 5 which we just proved,

$$\frac{d}{d\theta} \left[ \theta \mu^*(\hat{\theta}) - p^*(\hat{\theta}) \right] = (\theta - \hat{\theta}) \mu''(\hat{\theta}) - I(\hat{\theta}) > 0, \forall \hat{\theta} \leq \theta.$$

So if a buyer of type $\theta$ reports $\hat{\theta}$ such that $\hat{\theta} \leq \theta$, marginally increasing his report (and then always buying) can improve his payoff. \qed

**Claim 4.** For any type of buyer $\theta > \theta_1$, if he reports truthfully, then it is optimal to always follow the recommendation.
Proof. Fix any \( \theta > \theta_1 \), and suppose that he reports truthfully. As \( V(\theta) = \int_{\theta_1}^\theta D_1^*(s, s)ds > 0 \), he is willing to buy when recommended so. Meanwhile, at the cutoff value of \( \epsilon = m(\theta) \), the buyer’s payoff from buying is

\[
\theta (\mu^*(\theta) + m(\theta)) - p^*(\theta) = -\frac{\int_{m(\theta)}^\theta \theta [1 - F(\epsilon)] d\epsilon - \int_{\theta_1}^\theta D_1^*(s, s)ds}{1 - F(m(\theta))} < 0,
\]

where the equality follows from equations (20), (44) and (45), and the inequality follows from condition (47). This implies that at any \( \epsilon < m(\theta) \) that leads to the recommendation of not buying, it is optimal for the buyer not to buy. So on average, when receiving the recommendation of not buying, the buyer finds it optimal to follow. \( \square \)

Proof of Theorem 5. We now argue that the mechanism described in Theorem 5 satisfies (IC-quality). In particular, we are done if we can show that

\[
V(\theta) \geq \max_{\hat{\theta}} \theta \mu^*(\hat{\theta}) - p^*(\hat{\theta}), \forall \theta > \theta_1.
\]

Fix any \( \theta > \theta_1 \), and let \( \hat{\theta}^* \) be the maximizer of the RHS. By Claim 3, \( \hat{\theta}^* > \theta \). By Claim 4, it is optimal for type \( \hat{\theta}^* \) to always follow the recommendation (if he reports truthfully). This implies that

\[
\theta \left[ \mu^*(\hat{\theta}^*) + \mathbb{E}[\epsilon | \epsilon \leq m(\hat{\theta}^*)] \right] - p(\hat{\theta}^*) < \hat{\theta}^* \left[ \mu^*(\hat{\theta}^*) + \mathbb{E}[\epsilon | \epsilon \leq m(\hat{\theta}^*)] \right] - p(\hat{\theta}^*) \leq 0,
\]

where the first inequality follows from \( \hat{\theta}^* > \theta \), and the second inequality follows from the optimality of type \( \hat{\theta}^* \) to follow the (negative) recommendation (Claim 4). So after reporting \( \hat{\theta}^* \), it is never optimal for type \( \theta \) to disobey the recommendation of not buying. Therefore,

\[
\max_{\theta} \theta \mu^*(\hat{\theta}) - p^*(\hat{\theta}) \leq \max_{\hat{\theta}} \{0, U(\theta, \hat{\theta})\} \leq V(\theta),
\]

as desired. \( \square \)

Proof of Proposition 6. Suppose that \( \frac{1 - G(\theta)}{g(\theta)} \) is convex in \( \theta \). Recall that the pricing schedules with and without information design are

\[
p^*(\theta) = \frac{\int_{m(\theta)}^\theta [\theta (\mu^*(\theta) + \epsilon)] dF(\epsilon)}{1 - F(m(\theta))} - \frac{\int_{\theta_1}^\theta D_1^*(s, s)ds}{1 - F(m(\theta))},
\]

\[
p^*_{MR}(\theta) = \theta \mu^*(\theta) - \int_{\theta_{MR}}^\theta \mu^*(s)ds.
\]

When there is information design, let \( \theta_2 \) be such that \( m(\theta_2) = \xi \). That is, \( \theta_2 \) is the type above which the buyer is always recommended to buy.

By Proposition 5, for all \( \theta \in [\theta_{MR}, \theta_2] \),

\[
p^*(\theta) = \theta \mu^*(\theta) + I(\theta) < \theta \mu^*(\theta) = p^*_{MR}(\theta);
\]

\( ^{33} \) Any type \( \theta \leq \theta_1 \) is never recommended to buy. To ensure obedience, the seller can simply set a very high price.
for $\theta > \theta_2$.

$$p^*(\theta) = \theta \mu^*(\theta) = p^*_M(\theta).$$

Therefore, it is sufficient to show that $p^*(\theta_2) \geq p^*_M(\theta_2)$.

Toward proving this inequality, note that

$$p^*(\theta_2) = \int_{\theta_1}^{\theta_2} p^*(s)\,ds = \int_{\theta_1}^{\theta_2} \left\{ \int_{m(s)}^{\theta} [\mu^*(s) + \epsilon] \,dF(\epsilon) \right\} \,ds$$

where the second line follows from conditions (44) and (45). Therefore, it is sufficient to show that $p^*(\theta_2) \geq p^*_M(\theta_2)$.

Let us examine the concavity of $\int_{m^{-1}(\epsilon)}^{\theta_2} [\mu^*(s) + \epsilon] \,ds$ in $\epsilon$. First, writing $m^{-1}(\epsilon)$ as $\theta(\epsilon)$,

$$\frac{d}{d\epsilon} \left[ \int_{m^{-1}(\epsilon)}^{\theta_2} [\mu^*(s) + \epsilon] \,ds \right] = \theta_2 - m^{-1}(\epsilon) - [\mu^*(m^{-1}(\epsilon)) + \epsilon] \frac{1}{m'(m^{-1}(\epsilon))}$$

$$= \theta_2 - \theta - \frac{[\mu^*(\theta) + m(\theta)]}{m'(\theta)}$$

$$= \theta_2 - \theta - c(\mu^*) \frac{\left( \frac{1}{g(\theta)} \right)' - \frac{1}{g(\theta)}}{1 - \left( \frac{1}{g(\theta)} \right)'},$$

where the third line follows from conditions (44) and (45). Therefore,

$$\frac{d^2}{d\epsilon^2} \left[ \int_{m^{-1}(\epsilon)}^{\theta_2} [\mu^*(s) + \epsilon] \,ds \right] = \frac{d}{d\epsilon} \left[ \theta \left( \frac{1-G(\theta)}{g(\theta)} \right)' \frac{1}{1 - \left( \frac{1}{g(\theta)} \right)'^2} \right]$$

$$= \frac{\left( \frac{1}{g(\theta)} \right)''}{1 - \left( \frac{1}{g(\theta)} \right)'^2} \frac{d\theta}{d\epsilon} \leq 0,$$
where the inequality follows from \( \left( \frac{1-G(\theta)}{g(\theta)} \right)'' \geq 0 \) (convexity of \( \frac{1-G(\theta)}{g(\theta)} \)) and \( \frac{d\theta}{d\epsilon} = \frac{1}{m'(m^{-1}(\epsilon))} < 0 \). By Jensen’s inequality,

\[
E_\epsilon \left[ \int_{m^{-1}(\epsilon)}^{\theta_2} [\mu^*(s) + \epsilon] ds \right] \leq \int_{m^{-1}(0)}^{\theta_2} [\mu^*(s) + \epsilon] ds,
\]

so that \( p^*(\theta_2) \geq p^*_{MR}(\theta_2) \), as desired.

\[\square\]

### A.5 Optimal Mechanism with Binary Types and States

The derivation includes multiple claims and lemmas, and is more tedious than the continuous case. This section of the Appendix is intentionally omitted due to its excessive length; it is readily available upon request.
References


