The Dual Role of Ratings

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January 27, 2019

Abstract

Raters often collect payments from their ratees. In many markets, the ratee-pays structure sparks controversy on the conventional wisdom that ratings exist to solve adverse selection and moral hazard problems in trades. I prove that ratings tailored to maximize ratees’ payments fully solve moral hazard by leveraging the presence of adverse selection over time. These optimal ratings are coarse and opaque. I find a tension between rating transparency and economic efficiency, illustrate the implications of optimal ratings for market beliefs and behaviors, and reconcile the conventional wisdom with critiques that ratings add little information to the markets.

Keywords:
ratings, rating systems, repeated games, reputation, information intermediation.

JEL Classifications:
C72, C73, D82, D83, M52, G24.

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1 Introduction

Information intermediaries are central to markets with adverse selection and moral hazard problems. Examples abound, including intermediaries who rate borrowers (e.g., credit rating agencies), doctors (e.g., RateMD), employers (e.g., Glassdoor), hotels (e.g., TripAdvisor), restaurants (e.g., Yelp) or businesses in general (e.g., Better Business Bureau). The signals they produce, broadly conceived as ratings, coordinate market beliefs of the ratees’ abilities and their quality provisions over time. A conventional wisdom for why these ratings exist is their dual role to solve adverse selection and moral hazard problems in trades (e.g., see Dellarocas (2005), Gonzalez et al. (2004), Portes (2008) and Levich et al. (2012)).

Yet, these intermediaries often collect upfront payments from their ratees. The “ratee-pays” structure has sparked controversy on the conventional wisdom in many markets, including those listed above. Specifically, it is unclear whether rating schemes tailored to maximize ratees’ payments promote consumer learning and coordinate efficient trades, and failing to do so can impose a severe cost of resource misallocation.

The different goals are delicately intertwined. Transparent ratings tackle adverse selection but limit inept ratees’ willingness to pay, and may also fail to tackle moral hazard when competent ratees successfully build their reputations and rest on their laurels. In contrast, obfuscated ratings limit competent ratees’ willingness to pay. Ratings that fail to address either friction limit consumers’ payments to the ratees and in turn the ratees’ payments to the intermediary. Several questions arise: Whether and how do revenue-maximizing, ratee-pays ratings address adverse selection and moral hazard? What are their implications for market beliefs and behaviors? What is the structure of these ratings, and how does it depend on the market conditions?

I develop a simple model in Section 2 to address these questions. It captures the essential features of most ratee-pays rating relationships, without aspiring to describe closely any specific market. A firm repeatedly trades with a succession of short-lived consumers. Adverse selection and moral hazard problems are present: the firm has private information about its ability and choices of quality provision, and faces a myopic temptation to shirk against consumers. The firm may pay an upfront fee

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to participate in a rating scheme that rates the firm over time based on its track record. A rating quantifies the firm’s reputation and links its past behavior to market expectation of its future behavior. Thus, ratings act as a screening device and a commitment device that allow the rater to solve adverse selection and moral hazard respectively. The rater commits to a menu of rating schemes that maximizes her revenue via the upfront fees. I discuss and justify these features as the paper proceeds.

Section 3 collects the main results. I derive necessary and sufficient conditions for a menu of rating schemes to be optimal for the rater. Most importantly, optimal ratings leverage the presence of adverse selection to fully solve moral hazard. To illustrate, I explicitly characterize an optimal menu. The menu attracts participation by firms capable and incapable of high quality provision, maintaining market uncertainty over a rated firm’s ability. The ratings are coarse and opaque to fine-tune market beliefs, ensuring that reputation effect remains operative over time to overcome moral hazard. The ratings make the firm as valuable to consumers as it can possibly be, maximizing its willingness to pay the rater. Moreover, optimal ratings may contain no information absent moral hazard, but are maximally transparent absent adverse selection. I also find that the rater strictly benefits from the presence of adverse selection.

I show that optimal menus are socially efficient, illustrating a tension between rating transparency and economic efficiency. The results thus shed light on implications under a benevolent rater, such as those who derive revenues through commission fees (e.g., Expedia for accommodations and UpWork for freelancers) or advertising (e.g., Avvo for lawyers, RateMyProfessors for professors or review blogs with Google AdSense), as maximizing social welfare maximizes their popularity to facilitate trades.

I place the contributions of the results to the literature as the paper proceeds. To highlight the implications of ratings, I contrast the results to canonical repeated game settings without intermediation under pure moral hazard (Fudenberg and Levine, 1994), pure adverse selection (Tadelis, 1999) and both (Mailath and Samuelson, 2001). I identify new rationales behind the prevalence of coarse and opaque ratings, which is a common phenomenon in practice and a central question to the information intermediation literature since Lizzeri (1999). The results also reconcile the conventional wisdom with recent critiques on credit ratings suggesting that they add little information to the markets (Macey, 2006, Fitzpatrick and Sagers, 2009, Rhee, 2015), as the suppression of information flow is key to coordinating efficient trades.
2 Model

Consider a market populated by a long-lived firm, a sequence of short-lived consumers and a long-lived rater. Time is discrete, indexed by $t$, and the horizon is infinite.

The firm is one of two types $\theta$, competent ($C$) or inept ($I$). In each period, a competent firm chooses high ($\bar{e}$) or low effort ($\underline{e}$), and an inept firm only exerts low effort. High effort entails a cost $c > 0$, yielding good quality ($\bar{y}$) with probability $1 - \rho \in (\frac{1}{2}, 1)$ and bad quality ($y$) with probability $\rho$.\(^2\) Low effort is costless, yielding good quality with probability $\rho$ and bad quality with probability $1 - \rho$.\(^3\)

In each period, a consumer enters the market and pays the firm her expected payoff given her information, before the firm exerts effort. Good quality gives her a payoff of 1. Bad quality gives 0. I assume $1 - \rho - c > \rho$, so that high effort is efficient.

Before the first consumer enters, the rater offers the firm a menu of rating schemes $\Xi = \{\xi_C, \xi_I\}$. The firm chooses once-and-for-all whether to participate in a scheme and which scheme.\(^4\) The scheme $\xi_\theta = (f_\theta, R_\theta, S_\theta)$, intended for a firm of type $\theta$, consists of a fee $f_\theta \in \mathbb{R}_+$, a countable set of possible ratings $R_\theta$, and a rating system $S_\theta$ that maps each history of ratings and qualities to a probability distribution on $R_\theta$. The system $S_\theta$ draws a rating from $R_\theta$ before a consumer enters in each period. The fee $f_\theta$ specifies an upfront payment made to the rater by the firm who participates in $\xi_\theta$.

The calendar time and the menu are commonly observed by all parties. The type and effort choices are the firm’s private information. The firm also observes its participation decision, the scheme it chooses, and the histories of ratings and qualities. Each consumer also observes the rating in the period she enters and the quality before leaving the market, but no past history.\(^5\) The rating is either drawn by the system in a selected scheme, or is a null rating $\emptyset$ if the firm does not participate. Consumers and the rater share a common prior $\mu \in (0, 1)$ that the firm is competent. Each consumer uses the observed rating to update her belief of the firm being competent via Bayes’ rule, interpreted as the firm’s reputation. The rater also observes the firm’s participation decision, the scheme chosen, and the histories of ratings and qualities.

\(^2\)While the focus is on imperfect monitoring of the firm’s effort choices by the rater, the perfect monitoring case $\rho = 0$ is examined in Supplementary Appendix C.

\(^3\)Symmetry can be relaxed without affecting the results at the cost of additional notation.

\(^4\)The results are unaffected if the firm can quit the scheme after each history of play, at a significant notational cost. See Remarks 7 and 8 for a further discussion.

\(^5\)Whether the consumer observes the realized quality before leaving is formally irrelevant. Allowing so invites an interpretation that the consumer observes the quality and reports to the rater.
2.1 The Game

Each menu induces a game that unfolds as follows. In period $t = -1$, nature draws the firm’s type, choosing “competent” with probability $\mu$ and “inept” with probability $1 - \mu$. The firm learns its type before making a participation decision. If it chooses a scheme $\xi_\theta$, it pays the rater a fee $f_\theta$ in period $t = 0$. In each period $t = 0, 1, \ldots$, the entering consumer observes a rating and pays the firm. The firm exerts effort and generates a quality. The consumer leaves the market and period $t + 1$ unfolds. The moves are summarized in Figure 1 below.

Figure 1: Timing of Moves in a Period

I turn to define histories and strategies. Let $\Theta := \{C, I\}$ be the set of types, $D = \{N, \xi_C, \xi_I\}$ be the set of participation decisions where $N$ denotes not participating (i.e., the outside option), $A := \{\bar{e}, \bar{e}\}$ be the set of efforts, $Y := \{\bar{y}, \bar{y}\}$ be the set of qualities, and $R := R_C \cup R_I \cup \{\varnothing\}$ be the set of all ratings. The system’s period-$t$ history, denoted by $h_{1t}$, belongs to the set $H_{1t} = (Y \times R)^t$. It is the firm’s track record, consisting of previous qualities and ratings. The firm’s period-$t$ history upon observing the period-$t$ rating, denoted by $h_{2t}$, belongs to the set $H_{2t} = D \times (A \times Y \times R)^t \times R$ and consists of previous efforts, qualities and ratings. A period-$t$ consumer’s history upon observing the period-$t$ rating $r_t$ is simply $r_t$. For $i = 1, 2$, write $H_i := \cup_{t=0}^\infty H_{it}$.

A rating system in a scheme $\xi_\theta$ is a function $S_\theta : H_1 \to \Delta(R_\theta)$, which draws a rating $r_t$ with probability $S_\theta(r_t|h_1)$ after each history $h_1$.

A firm’s strategies are a pair $\sigma = (\pi, \tau)$. The participation strategy $\pi : \Theta \to \Delta(D)$ specifies $\pi(d|\theta)$, the probability of choosing participation $d \in D$ by each type $\theta$. The effort strategy $\tau : H_2 \times \Theta \to [0, 1]$ specifies the probability of high effort after each history $h_2$ for each type, with the restriction $\tau(\cdot, I) = 0$.

Consumers’ beliefs are defined over the set of outcomes of the game, $\Omega := \Theta \times D \times (A \times Y \times R)^\infty$. Given $\mu, \sigma$ induces a probability measure $P \in \Delta(\Omega)$. Let $P_\theta^\epsilon$
denote the marginal of $P$ on $H_1$ conditional on $\theta$. The probability $P_t^\theta(r|d)$ of realizing a rating $r \in R$ in period $t$ given that a type-$\theta$ firm chooses participation $d \in D$ is

$$P_t^\theta(r|\xi_{\theta'}) := \sum_{h_t^1 \in H_t^1} S_{\theta'}(r|h_t^1)P_t^\theta(h_t^1|\xi_{\theta'}) \text{ and } P_t^\theta(\emptyset|N) = 1,$$

where $P_t^\theta(h_t^1|\xi_{\theta'})$ is the probability of realizing a history $h_t^1$ in period $t$ given that the firm has a type $\theta$ and chooses $\xi_{\theta'}$. A period-$t$ consumer who observes a rating $r$ forms a posterior of the firm being competent:

$$\varphi_t(r) = \frac{\mu \sum_{d \in D} \pi(d|C)P_t^C(r|d)}{E^\mu[\sum_{d \in D} \pi(d|\theta)P_t^\theta(r|d)]},$$

where $E^\mu[\cdot]$ is an expectation over types $\theta$ with respect to $\mu$. A period-$t$ consumer's belief is a function $p_t^\sigma : R \to [0, 1]$ capturing the probability of receiving a good quality upon observing a rating, and equals her payment given the payoff normalizations.

The firm, with a discount factor $\delta \in (0, 1)$, chooses $\sigma$ to maximize its payoff. A type-$\theta$ firm extracts a normalized profit from consumers upon participation $d$ equal to

$$U(\sigma, \theta; d) := (1 - \delta)E^P\left[\sum_{t=0}^{\infty} \delta^t (p_t^\sigma(r_t) - c(e_t)) \bigg| \theta, d\right],$$

where $c(e_t)$ denotes the cost incurred by choosing effort $e_t$ in period $t$, and the expectation $E^P[\cdot|\theta, d]$ is taken with respect to the measure $P$ induced by $\sigma$, conditional on $\theta$ and $d$. The firm’s payoff is its profit minus any participation payment:

$$U^*(\sigma, \theta) := \sum_{d \in D} \pi(d|\theta)U(\sigma, \theta; d) - (1 - \delta) \sum_{\theta'} \pi(\xi_{\theta'}|\theta)f_{\theta'},$$

Equilibrium refers to perfect Bayesian equilibrium $(\sigma, \varphi)$ such that $\sigma$ is maximizing after each history of the firm of each type given beliefs $\varphi = (\varphi_t)_{t=0}^{\infty}$, and the beliefs are consistent with Bayes’ rule given $\sigma$ whenever possible.

### 2.2 The Rater’s Problem

The rater seeks a revenue-maximizing menu. She has commitment, in the sense that the menu is chosen once and for all. For each menu $\Xi$, denote the set of all equilibria
in the induced game by $B(\Xi)$. The rater’s normalized payoff using a menu $\Xi$ in an equilibrium $(\sigma, \varphi) \in B(\Xi)$ is

$$W(\Xi, (\sigma, \varphi)) := (1 - \delta)E^\mu \left[ \sum_{\theta' \in \Theta} \pi(\xi_{\theta'} \mid \theta) f_{\theta'} \right]. \quad (4)$$

The focus is on rater-preferred equilibria, those that generate the highest rater’s payoff. The rater’s problem is therefore

$$\sup_{\Xi} \sup_{(\sigma, \varphi) \in B(\Xi)} W(\Xi, (\sigma, \varphi)). \quad (5)$$

A menu $\Xi$ is optimal if it attains the supremum in (5) as a maximum. I close the section with several remarks on the essential features of the model.

**Remark 1** (Prices). The familiar assumption that consumers pay their expected payoffs (Holmström, 1999, Mailath and Samuelson, 2001, Board and Meyer-ter Vehn, 2013) permits a focus on the strategic interaction between the rater and the firm. Upon observing a rating $r$ in an equilibrium $(\sigma, \varphi)$ in period $t$, a consumer pays

$$p^\sigma_t(r) = \varphi_t(r)E^P[\tau(h^t_{C, r}) \mid r_t = r, \theta = C](1 - \rho)$$

$$+ (1 - \varphi_t(r)E^P[\tau(h^t_{C, r}) \mid r_t = r, \theta = C])\rho$$

$$= \rho + (1 - 2\rho) \varphi_t(r) E^P_\text{reputation}[\tau(h^t_{C, r}) \mid r_t = r, \theta = C]. \quad (6)$$

The conditional expectation of the competent firm’s effort is taken over possible firm’s histories $h^t_2$ since the consumer is unaware of the precise history observed by the firm except the rating $r_t = r$. Expression (6) highlights the role ratings play in coordinating consumers’ beliefs of the firm’s type and effort choices. It describes formally how ratings affect consumers’ beliefs and the firm’s revenue, and in turn the firm’s payment to the rater. To maximize her payoff, the rater wishes to maximize both the firm’s reputation and its expected effort in each period.

**Remark 2** (Market structure). The market structure is familiar from the literature on seller reputation. There is adverse selection because consumers (and the rater) are unsure of the firm’s type. Moral hazard arises because the competent firm faces a
myopic temptation to shirk against consumers. Absent intermediation, the competent firm would shirk upon receiving the upfront payment, because consumers do not observe any past history. Each consumer thus pays the firm $\rho$ in equilibrium. However, the firm would obtain a higher profit if it could convince consumers that it is competent and that it could commit to exert high effort. Thus, by adequately revealing past information to consumers, ratings can act as a screening device to solve the adverse selection problem and as a commitment device to solve the moral hazard problem.

Finally, a Stackelberg type of firm who only exerts high effort is absent, contrary to canonical reputation models (Fudenberg and Levine, 1989, 1992). The presence of an inept type, however, is particularly appealing for the markets mentioned in the outset that motivate the paper: consumers often approach these markets not to find a firm bound to deliver good quality, but to avoid the firms incapable of doing so.

Remark 3 (The rating relationship). The upfront fee captures in a simple manner the source of a rater’s revenue under the ratee-pays paradigm, and is familiar from the literature (e.g., Lizzieri (1999) and Farhi et al. (2013)). In practice, this monetary transfer from the ratee to the rater may involve a stream of constant payments fixed ex ante. The fee in the model can be viewed as a discounted sum of these payments.

The once-and-for-all participation decision captures the prevalent feature that most rating relationships are long-lived due to their impact on both parties’ profits, and termination of the relationship is rare.8

Following the lead of the information intermediation literature, I also assume that the rater has full commitment power.9 This assumption is plausible when commitment arises from reputational concern of the intermediary.10 In the model, the fact that the rating system maps to a probability distribution over ratings might make the

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8Partnoy (2006) documents that the three largest credit rating agencies, Moody’s, Standard & Poor’s and Fitch, receive around 90% of their revenues from fees paid by issuers. A withdrawal of the rating relationship has severe adverse impact on the firm’s profit extraction from consumers. See Salvadoré (2014, 2017) for related documents in the context of restaurant ratings on Yelp, and Fickenscher (2017) in the context of the hotel industry.


10The rater is viewed as a “reputational intermediary,” who strives to publish credible signals to the market (Coffee, 1997). A famous example of a rater’s desire to signal its commitment is the expulsion of the Los Angeles affiliate of Council of Better Business Bureaus, after it was discovered in 2009 that several eateries in Southern California simply paid for high ratings (Better Business Bureau, 2013). In the context of credit ratings, the raters often disclose information regarding their rating methodologies ex ante and their rating processes are monitored by the U.S. Securities and Exchange Commission who publishes public annual reports regarding their performances.
assumption appear tenuous, since a deviation is presumably harder to detect than if the system maps to deterministic ratings. In fact, the optimal rating systems characterized in Section 3 are deterministic.

**Remark 4** (Full transparency). The rater can recover a standard repeated game setting using a *fully revealing* rating system. A rating system is fully revealing if each consumer puts probability one on the true history of qualities upon seeing each rating from the system. The standard repeated game counterpart to this paper is Mailath and Samuelson (2001), the relation to which is discussed extensively in Section 3.5.

### 3 Results

#### 3.1 General Characterization

The first main result collects the necessary and sufficient conditions for a menu to be optimal. The result rests upon the following assumption, which is a sufficient condition for ratings to be capable of generating the maximal market surplus.

**Assumption 1** (Effective Intermediation). The parameters $\mu, \delta, \rho, c$ satisfy

$$c \leq \delta(1-2\rho)^2 \left( \frac{1-\mu}{1-\mu+\mu\rho} \right) =: \bar{c}(\mu, \delta, \rho).$$

(7)

Condition (7) is a joint market condition that refines the static, benchmark condition for efficiency of high effort, namely $c < 1 - 2\rho$. The static condition is bound to be ineffective in a dynamic environment that features adverse selection and moral hazard. To motivate high effort, incentives must be strong enough to ensure that the benefit of high effort outweighs the effort cost. The benefit, however, is only partially captured through future consumer payments due to discounting and incomplete information over the firm’s type while the cost is incurred immediately.

Assumption 1 is weaker when the firm is relatively patient (high $\delta$), or if monitoring is relatively precise (small $\rho$), or if consumers are skeptical of the firm’s ability of quality provision (small $\mu$). It invites an interpretation that intermediation is effective for coordinating beliefs and behaviors in the market, making it quite plausible for the research question at hand. One would expect ratings to be capable of disciplining a

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11Appendix A.1 provides details on the construction of a fully revealing rating system.
firm’s behavior when the firm cares about its future profits influenced by the ratings, and when monitoring is less noisy. Finally, ratings add little to affecting consumers’ beliefs when consumers are born confident about the firm being competent.

**Proposition 1** (Necessary and Sufficient Conditions for Optimal Menus). *Under Assumption 1, a menu \( \Xi \) is optimal if and only if there exists an equilibrium \((\sigma, \varphi) \in B(\Xi)\) such that*

1. \( \pi(\Xi|C) = 1 \),
2. \( \pi(\Xi|I) > 0 \),
3. \( \tau(h_2, C) = 1 \) after every history \( h_2 \) that occurs with positive probability conditional on the firm’s type being competent,
4. for each type \( \theta \), if \( \pi(\xi_\theta|\theta) > 0 \), then \( (1 - \delta)f_\theta = U(\sigma, \theta; \xi_\theta) - \rho \).

*The rater’s optimal payoff is \( \mu(1 - 2\rho - c) \).*

The first condition says that the competent type participates with probability one. Second, the inept type participates with positive probability. Third, the competent firm consistently exerts high effort upon participation. Finally, if a firm participates in a scheme with positive probability, then the fee extracts all its profits above \( \rho \).

The proof is in Appendix B.2. It first establishes an upper bound on the rater’s expected payoff. Plainly, it is the maximal discounted sum of expected profits she extracts from a firm in period 0. For participating in a scheme to be individually rational, the firm must obtain a payoff of at least \( \rho \) by participating. This is the lowest payoff a firm can guarantee by choosing the outside option, since each consumer pays at least \( \rho \) by (6). Moreover, the expected probability of a firm being competent is simply \( \mu \), due to the martingale property of posterior beliefs. Because high effort is efficient, the maximal expected surplus that can be extracted by the rater in each period is \( \mu(1 - \rho - c) + (1 - \mu)\rho \). The rater’s expected payoff is therefore at most

\[
(1 - \delta) \sum_{t=0}^{\infty} \delta^t [\mu(1 - \rho - c) + (1 - \mu)\rho] - \rho = \mu(1 - 2\rho - c).
\]

The proof then shows that the rater obtains this payoff in an equilibrium using some menu if and only if several conditions hold. First, to generate the maximal surplus \( 1 - \rho - c \) in the event that the firm is competent, equilibrium calls for consistent high effort by the competent type upon participation.
Equilibrium consistent high effort in turn requires attracting each type to participate with positive probability. Suppose on the contrary that there is an equilibrium in which only the competent firm participates with positive probability, and it consistently exerts high effort. The competent firm’s reputation equals one upon participation and consumer payments always equal $1 - \rho$ by (6). This destroys the putative equilibrium, as the competent firm faces an irresistible temptation to shirk after each history upon participation, saving on effort cost yet receiving the same future revenue.

Moreover, the rater to extract a profit from the competent firm with probability equal to $\mu$ if and only if the competent firm participates with probability one.

The proof concludes by showing that a menu $\Xi^*$ and an equilibrium $(\sigma^*, \varphi^*)$ in the induced game give the rater the payoff $\mu(1 - 2\rho - c)$ under Assumption 1. Any menu that gives the rater a payoff short of this value must be sub-optimal. The construction of the menu $\Xi^*$ and the equilibrium $(\sigma^*, \varphi^*)$ is described in detail in the next section.

Remark 5 (The value of adverse selection). It is intuitive that when the prior $\mu$ is relatively large, the ability to induce consistent high effort by attracting both types is a virtue for the rater, because consumer payments are always close to the maximal payment $1 - \rho$ by (6). This raises the firm’s expected profits and thus the rater’s revenue. However, the idea that attracting both types is necessary for optimality for every $\mu \in (0, 1)$ may appear surprising at first glance. When $\mu$ is small, consumers consider the firm likely to be inept and pay a smaller amount in view of (6) despite consistent high effort. A reasonable conjecture is that the rater wishes to set a sufficiently high rating fee that screens away the inept type (i.e., rules out equilibria in which the inept type participates with positive probability), raising consumers’ beliefs of a participating firm’s type and their willingness to pay. Lemma 2 in Appendix B shows precisely that this seemingly compelling intuition is wrong. Specifically, in any equilibrium in which only the competent firm participates with positive probability, the rater obtains at most

$$\mu \left( 1 - 2\rho - c - \frac{\rho c}{1 - 2\rho} \right) < \mu(1 - 2\rho - c).$$

Screening away the inept firm suppresses the rater’s expected payoff via two channels when $\mu$ is small. First, consistent high effort is no longer possible. Second, the rater considers the firm unlikely to be competent, and thus unlikely to collect the rating fee.
Remark 6 (The value of intermediation). Because optimal menus fully extract a participating firm’s profits above $\rho$, the firm obtains the same payoff as if no intermediation is available. The important difference in the two settings rests upon the firm’s behavior and hence the firm’s value to consumers. Under optimal intermediation, the competent type consistently exerts high effort and is as valuable to consumers as it can be. Importantly, because by construction consumers always get an expected payoff of 0 and the payment to the rater by the firm is simply a monetary transfer, optimal menus are socially efficient, despite the self-interested rater.

3.2 An Optimal Menu of Rating Schemes

I now describe the menu $\Xi^* = \{\xi^*_C, \xi^*_I\}$ and the equilibrium $(\sigma^*, \varphi^*)$ constructed to prove Proposition 1. The construction illustrates explicitly how optimal ratings leverage the presence of adverse selection to fully solve the moral hazard problem. In the scheme $\xi^*_C$, the rating set $R^*_C = \{0, 1\}$ is binary. The rating system $S^*_C$ announces a rating 1 if the most recent quality is good, and announces a rating 0 otherwise:

$$S^*_C(h_1^C) = \begin{cases} 1 \circ \{1\}, & \text{if } y_{t-1} = \bar{y}, \\ 1 \circ \{0\}, & \text{otherwise.} \end{cases} \quad (8)$$

In the scheme $\xi^*_I$, the rating set $R^*_I = \{0\}$, and the system $S^*_I$ always gives a rating 0:

$$S^*_I(h_1^I) = 1 \circ \{0\}, \text{ for every } h_1. \quad (9)$$

The fees $f^*_C$ and $f^*_I$ are given in Appendix A.2. They extract all profits above $\rho$ of the participating firm in the equilibrium $(\sigma^*, \varphi^*)$.

The equilibrium $(\sigma^*, \varphi^*)$, where $\sigma^* = (\pi^*, \tau^*)$, specifies that the firm participates with probability one in the scheme intended for its type in the menu $\Xi^*$, and the competent type always exerts high effort upon choosing the “competent” scheme:

$$\pi^*(\xi^*_C|C) = 1, \quad \pi^*(\xi^*_I|I) = 1, \quad (10)$$

$$\tau^*(h_2, C) = \begin{cases} 1, & \text{if } d = \xi^*_C, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

Consumers believe that a firm who deviates to choose the outside option is inept for
sure, $\varphi^*_t(\emptyset) = 0$ for every $t$. By (6), the firm’s outside option payoff is precisely $\rho$.

**Corollary 1 (Optimal Rating Menu).** *Under Assumption 1, the rating menu $\Xi^*$ is optimal, and $(\sigma^*, \varphi^*) \in B(\Xi^*)$ is a rater-preferred equilibrium.*

By (6), prices satisfy $p^*_t(r) = \rho + (1 - 2\rho)\varphi^*_t(r)$ in the equilibrium. Each consumer puts probability one on the firm being competent upon observing the rating 1, because it must be drawn from the system $S^*_C$. This leads to the maximal payment $p^*_t(1) = 1 - \rho$ for any period $t \geq 1$. In contrast, the rating 0 is *opaque* in the sense that consumers cannot infer the firm’s type via the rating. In any period $t \geq 1$, an entering consumer who observes a rating 0 infers that either the firm is competent but delivers a bad quality in the last period or the firm is inept. Consumer payment then equals

$$p^*_t(0) = \rho + (1 - 2\rho)\varphi^*_t(0) = \rho + (1 - 2\rho)\left(\frac{\mu\rho}{\mu\rho + 1 - \mu}\right).$$

The rating 0 in period 0 does not reflect any past quality, thus $\varphi^*_0(0) = \mu$, and consumer payment equals $p^*_0(0) = \rho + (1 - 2\rho)\mu$.

The competent firm exerts high effort after every history upon participating in $\xi^*_C$, in chase of a high consumer payment associated with rating 1 and to avoid the low payment associated with rating 0 in each period. Because the system $S^*_C$ has one-period memory, the discounted benefits of high effort in any period $t$ realized in period $t + 1$ must outweigh the immediate cost incurred:

$$\delta\left[\frac{(1 - \rho)p^*_t(1) + \rho p^*_0(0)}{\text{expected payment following high effort}} - \frac{(\rho p^*_t(1) + (1 - \rho)p^*_0(0))}{\text{expected payment following low effort}}\right] \geq (1 - \delta)c.$$

Assumption 1 ensures that this incentive constraint for high effort is satisfied.

The rating system $S^*_C$ is *coarse*: only two ratings are sent in each period $t \geq 1$, despite a firm’s exponentially growing number of possible track records. The ratings limit consumer learning of a rated firm’s track record, but is crucial for fully solving moral hazard. They ensure that consumer posteriors are sufficiently responsive to every quality delivered by the firm, sustaining reputation effect on the incentives for high effort over the long run. To see this, consider the fully revealing system introduced in Remark 4. In an equilibrium in which both types are rated by this system, the competent type cannot consistently exert high effort. Otherwise, consumer posteriors would eventually become very close to 1 conditional on the firm being competent, and
further qualities delivered have virtually no impact on the posteriors. The competent
type then finds it profitable to shirk, destroying the putative equilibrium.

This observation illustrates a tension between rating transparency and economic
efficiency. A singleton menu that specifies a fully revealing rating system is sub-optimal
by Proposition 1, because moral hazard cannot be fully solved.

The opaqueness of rating \( 0 \) is also crucial for consistent high effort. If the rating \( 0 \)
in the inept scheme is instead labelled as \( 2 \), then both ratings \( 1 \) and \( 0 \) perfectly reveal
the firm’s type being competent. This effectively fully solves the adverse selection
problem. In view of Remark 5, the moral hazard problem cannot be fully solved.

Finally, together with the ratings, the fee structure ensures that the firm finds it
incentive-compatible to choose the scheme intended for its own type. Because the
competent type consistently exerts high effort, an inept type who deviates to participate
in the “competent” scheme obtains the higher payment \( p_{1}^{*} \) (1) less frequently and
thus obtains less expected profits than the competent firm does. This deviation must
give the inept firm a payoff strictly less than \( \rho \), because the competent firm obtains a
payoff of exactly \( \rho \) after paying the fee \( f_{c}^{*} \). On the other hand, the competent firm is
indifferent between both schemes. By deviating to the “inept” scheme, the competent
firm extracts the same profits from consumers as the inept type does because the
inept scheme \( \xi_{I}^{*} \) is fully coarse: there is only one rating and thus one possible payment.
Since the inept firm obtains a payoff \( \rho \) after paying the fee \( f_{I}^{*} \), so does the competent
firm upon the deviation.

A central theme of research in the information intermediation literature asks why
intermediaries often provide coarse ratings in practice, withholding some collected
information. The present result that the revenue-maximizing ratings are coarse and
opaque is familiar from Lizzeri (1999) in the context of static certification absent moral
hazard.\(^{12}\) Contrary to the certification literature, here the rater does not observe the
participating firm’s private type. The ratings thus serve a screening role, ensuring
that the firm picks the scheme intended for its own type.

The ratings also serve a sanctioning role, because they need to tackle moral hazard
and incentivize consistent high effort by the competent firm. To carry out this role,
the ratings censor past play from consumers to ensure reputation effect over the
competent firm’s effort choices remain effective over the long run. This property
of limited memory to maintain long-run reputation effect is familiar from Ekmekci

\(^{12}\)See also Harbaugh and Rasmusen (2018) and the references therein.
He characterizes a rating system with one-period memory that achieves frequent high effort by a long-lived firm in a repeated product-choice game, where the firm faces a succession of short-lived consumers and the firm might be a Stackelberg type. Contrary to the equilibrium he constructs which features low effort upon some ratings, the assumption here that the firm is possibly an inept type but never a Stackelberg type provides the competent firm with a perpetual incentive to exert high effort to “separate” from the inept type.

Remark 7 (Off-path belief). The belief of a non-participating firm being inept for sure in the equilibrium \((\sigma^*, \varphi^*)\) is stronger than necessary. In view of Remark 6, a non-participating firm finds it optimal to consistently exert low effort. Each consumer thus pays the firm \(\rho\), as if she believes the firm is inept for sure.

Remark 8 (The no opt-out assumption). Remark 7 implies that even if the model allows the firm to quit or suspend the rating relationship in the beginning of each period before a rating is realized, the firm never have a strict incentive to do so in any equilibrium given an optimal menu, as continuing allows the firm to be at least as well off. Any equilibrium in which the firm quits or suspends the relationship must not be rater-preferred, because the rater’s payoff falls short of the upper bound.

3.3 When and Why are Ratings Informative?

I now turn to show that the presence of moral hazard forces optimal ratings to contain some information content. I begin with a general characterization of optimal menus absent moral hazard \(i.e.,\) in a pure adverse selection setting, in which effort cost \(c = 0\) and the competent firm only exerts high effort, \(\tau(\cdot, C) = 1\).

Proposition 2 (Pure Adverse Selection). In a pure adverse selection setting, a menu \(\Xi\) is optimal if and only if one of the following conditions is true:

A. there exists an equilibrium \((\sigma, \varphi) \in B(\Xi)\) such that

1. \(\pi(\Xi|C) = 1\),

13Relatedly, see Liu (2011) and Liu and Skrzypacz (2014).

14In the equilibrium constructed, the firm exerts low effort yet consumers pay the firm a high price upon the best rating, giving the firm a high payoff. Upon the worst rating, however, the firm exerts low effort and consumers pay a low price, giving the firm a low payoff. The normal type of firm exerts high effort whenever other ratings emerge in an attempt to achieve the best rating.
2. \( \pi(\Xi|I) > 0 \),
3. for each type \( \theta \), if \( \pi(\xi_{\theta'}|\theta) > 0 \), then \((1 - \delta)f_{\theta'} = U(\sigma, \theta; \xi_{\theta'}) - \rho \).

B. there exists an equilibrium \((\sigma, \varphi) \in B(\Xi)\) such that

1. \( \pi(\Xi|C) = 1 \),
2. \( \pi(\Xi|I) = 0 \),
3. if \( \pi(\xi_{\theta}|C) > 0 \), then \((1 - \delta)f_{\theta} = U(\sigma, C; \xi_{\theta}) - \rho \).

The rater’s optimal payoff is \( W = \mu(1 - 2\rho) \).

The proof is in Appendix B.3. Not surprisingly, the set of optimal menus expands in comparison to Proposition 1, as the rater no longer needs to motivate high effort. The conditions in Part A are familiar from Proposition 1. The additional set of optimal menus, characterized by Part B, induces an equilibrium in which only the competent firm participates, and does so with probability one. In the schemes to which it assigns positive probability for participation, the fees fully extract its profits above \( \rho \).

I construct an optimal menu \( \Xi_{AS} = \{\xi_{AS}\} \) with an equilibrium satisfying the conditions in Part A, in which ratings contain no information content. The rating set contains only a rating 0, and the system always announces the rating 0. The fee \( f_{AS} \) satisfies \((1 - \delta)f_{AS} = \mu(1 - 2\rho) \). The equilibrium calls for participation by both types with probability one, and consumers believe that a firm with a null rating is inept for sure. This is an equilibrium because the fee \( f_{AS} \) binds the participation constraints of both types: analogous to Proposition 1, the expected market surplus in each period is \( \mu(1 - \rho) + (1 - \mu)\rho = \rho + \mu(1 - 2\rho) \), while the firm’s outside option payoff is \( \rho \). This menu and equilibrium thus exclude consumer learning and intermediation adds no value to consumers: it has no impact on consumers’ beliefs and payments in each period. When moral hazard is present, information must be revealed to create differential payments to sustain incentives for high effort.

There is another optimal menu \( \Xi'_{AS} = \{\xi'_{AS}\} \) with an equilibrium satisfying the conditions in Part B. In the scheme \( \xi'_{AS} \), the rating set contains only one non-null rating, and the system always announces that rating. The fee \( f'_{AS} \) satisfies \((1 - \delta)f'_{AS} = 1 - 2\rho \). In the equilibrium, only the competent firm participates, and does so with probability one. Neither type wishes to deviate from their participation choices because it would earn a payoff of \( \rho \) either way. An inept firm is immediately exposed to consumers once they observe a null rating and obtains \( \rho \). The competent firm obtains \( 1 - \rho \).
from consumers, and the fee $f'_{AS}$ extracts all its profit above $\rho$. The rater obtains an expected payoff $\mu(1 - 2\rho)$.

### 3.4 The Implications of Adverse Selection

This section illustrates the implications of adverse selection to the rater and the competent firm. To set the stage, I characterize an optimal rating menu in a pure moral hazard setting, in which the firm is commonly known to be competent, that is, $\mu = 1$. The characterization rests upon the following assumption, ensuring that ratings are capable of inducing the largest possible market surplus.

**Assumption 2.** The parameters $\delta, \rho, c$ satisfy

$$c \leq \frac{\delta(1 - 2\rho)^2}{1 - \delta\rho} =: \bar{c}_{MH}(\delta, \rho). \quad (12)$$

Not surprisingly, this assumption is weaker than Assumption 1, because the incomplete information over the firm’s type that depresses the benefits of exerting high effort is no longer present.

**Proposition 3** (Pure moral hazard). Consider a pure moral hazard setting. The competent firm’s profit extraction from consumers is bounded above by

$$1 - \rho - c - \frac{\rho c}{1 - 2\rho}. \quad (13)$$

The rater’s expected payoff is bounded above by

$$1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}. \quad (14)$$

Under Assumption 2, there exists an optimal menu $\Xi_{MH} = \{\xi_{MH}\}$ and an equilibrium in the induced game that give the rater exactly the payoff (14) and the competent firm exactly the profit (13) from consumers.

The construction of the menu $\Xi_{MH}$ and the equilibrium is relegated to Appendix A.3. Absent moral hazard, consistent high effort is not possible in equilibrium. Optimal ratings here reward the competent firm by revealing good qualities to motivate high
effort as frequently as possible.\textsuperscript{15} When there is adverse selection and hence the possibility to induce consistent high effort in equilibrium, the rater can achieve the maximal payoff $\mu(1 - 2\rho - c)$ by Proposition 1, exceeding (14) when $\mu$ is sufficiently large. In contrast, the competent firm is worse off in the presence of adverse selection:

**Proposition 4.** In the setting with both adverse selection and moral hazard, the competent firm’s equilibrium profit from consumers is strictly bounded above by (13).

Proposition 4 illustrates a contrast with canonical reputation games under imperfect monitoring in which the Stackelberg type can arise (Fudenberg and Levine, 1992). In those cases, the competent player subject to binding moral hazard may be assured a payoff in the game with incomplete information over the firm’s type in excess of any equilibrium payoff in the complete-information game. Incomplete information opens the possibility of a horizon during which the consumers are uncertain of the firm’s type, and in which the play does not resemble any equilibrium play of the complete-information game and is in favor of the competent firm. Here, the possible presence of an inept firm poses a negative impact on consumers’ payments and the competent firm’s profits.

### 3.5 The Implications of Optimal Ratings

I now turn to highlight the economic implications of optimal ratings for market beliefs and behaviors by relating the present results to the literature of repeated games. In the canonical settings, there are no ratings and no rater, and consumers observe the history of qualities. To make a reasonable comparison, I focus on the competent firm’s expected profit extraction from consumers.

#### 3.5.1 Adverse Selection and Moral Hazard: Rater versus No Rater

Consider first the main setting of interest in which both adverse selection and moral hazard are present. The corresponding canonical setting is Mailath and Samuelson (2001). In the proof of Proposition 1, I show that the competent firm’s expected profit

\textsuperscript{15}This result is familiar from Dellarocas (2005), who considers a product-choice game with pure moral hazard and finds that effort-maximizing ratings coordinate different effort choices in reward and punishment phases.
given the menu $\Xi^*$ in equilibrium $(\sigma^*, \varphi^*)$ is

$$\bar{U}_{\text{rating}}^{\text{AS+MH}}(\mu) := \rho + \mu(1 - 2\rho) + \frac{\delta(1 - \mu)^2(1 - \rho)(1 - 2\rho)}{1 - \mu(1 - \rho)} - c. \quad (15)$$

In the canonical setting, it is standard to show that the upper bound on a firm’s equilibrium profit is\(^{16}\)

$$\bar{U}_{\text{public}}^{\text{AS+MH}}(\mu) := \rho + (1 - \delta)(1 - 2\rho) \sum_{t=0}^{\infty} \frac{\delta^t(1 - \rho)^t\mu}{(1 - \rho)^t \mu + \rho^t(1 - \mu)} - c - \frac{\rho c}{1 - 2\rho}. \quad (16)$$

When the effort cost $c$ is sufficiently small, there exists an equilibrium in the canonical setting in which the competent firm frequently but never consistently exerts high effort (Mailath and Samuelson, 2015), giving the firm a profit close to (16).\(^{17}\)

The comparison between the two bounds is depicted in Figure 2 for

$$\mu \leq \bar{\mu}(c, \delta, \rho) := 1 - \frac{c\rho}{\delta - (c + 4\delta\rho)(1 - \rho)},$$

which is obtained by rearranging Assumption 1, with $c = 0.1, \delta = 0.9$ and $\rho = 0.25$. The effect of consistent high effort brought by optimal ratings is particularly dramatic when $\mu$ is sufficiently close to $\bar{\mu}$. In this event, the competent firm extracts a higher profit when the rater obscures information rather than publicly disclosing the complete history of qualities. Consumer payments remain relatively large during punishment phases (i.e., upon “bad” rating) because consumers’ posteriors are only slightly less than $\mu$, pushing the firm’s profit beyond the canonical bound. When $\mu$ is low, the competent firm manages to extract more profits from consumers than the canonical setting because the scheme $\xi^*_C$ reveals the competent firm’s true type immediately and perfectly every time after a good quality. In the canonical setting, it takes time for the competent firm to build up its reputation to enjoy high consumer payments under frequent high effort. For the intermediate values of $\mu$, the competent firm’s payoff upper bound in the canonical setting may exceed that in the rating case when the firm is relatively patient. In the canonical setting, the initial periods of relatively low payments then contribute less to the firm’s profits, and the later periods of higher consumer payments when it gradually builds up its reputation contribute more under

\(^{16}\)See Supplementary Appendix G for detailed calculations.

\(^{17}\)The failure to sustain consistent high effort here is an instance of a more general result due to Cripps, Mailath, and Samuelson (2004).
frequent high effort. When posteriors are large, a bad quality has little impact on the posterior. In the rating game, however, each bad quality leads immediately to the low payment associated with rating 0, regardless of other past performances.

\[ \bar{U}_{\text{public}}(\mu) := (1 - \delta)(1 - 2\rho) \sum_{t=0}^{\infty} \delta^t \frac{(1 - \rho)^t \mu}{(1 - \rho)^t \mu + \rho^t(1 - \mu)} + \rho. \]  

(17)

In the rating game with pure adverse selection, the optimal menu \( \Xi'_{\text{AS}} \) characterized in Section 3.3 induces an equilibrium that reveals the firm’s type to consumers upon participation. The competent firm thus extracts the maximal profit \( 1 - \rho \) in every period. For every \( \mu \in (0, 1) \), this once-and-for-all reputation establishment allows the firm to extract strictly higher profit, as illustrated in Figure 3 below.

**Figure 2:** Implications under both Adverse Selection and Moral Hazard

The infinite sum in (16) is approximated by the corresponding summation from \( t = 0 \) to \( t = 10000 \).

### 3.5.2 Pure Adverse Selection: Rater versus No Rater

Consider next the comparison under pure adverse selection. The canonical setting with pure adverse selection corresponds to the market interaction in Tadelis (1999). The equilibrium profit upper bound can be obtained by taking \( c = 0 \) in (16), giving

\[ \bar{U}_{\text{AS} + \text{MH}}(\mu; \xi^*) = \bar{U}_{\text{AS}}(\mu) + \frac{\rho}{1 - \rho}. \]
Figure 3: Implications under Pure Adverse Selection or Pure Moral Hazard

3.5.3 Pure Moral Hazard: Rater versus No Rater

Consider finally the comparison under pure moral hazard. Here, the canonical setting corresponds to Fudenberg and Levine (1994). The maximal equilibrium profits in the rating game and in the canonical setting must coincide, because maximizing the rating fee that can be extracted amounts to maximizing the competent firm’s profits. Proposition 3 implies that the profit upper bound in both settings is

\[
\bar{U}_{\text{.rating}}^{\text{MH}}(\mu; \xi_{\text{MH}}) = \bar{U}_{\text{public}}^{\text{MH}}(\mu) = 1 - \rho - c - \frac{\rho c}{1 - 2\rho},
\]

again illustrated in Figure 3. Further, the proposition shows that the bound is tight in the rating game using a scheme with two ratings, and transitions between the ratings across each period depends only on the most recent quality realized. Indeed, the bound is also tight in the canonical setting. One can interpret the rating system as a two-state automaton depicting an equilibrium with frequent high effort in the canonical setting, such that one rating corresponds to the “good” state in which the firm exerts high effort and obtains the maximal stage profit \(1 - \rho - c\), and the other rating corresponds to the “bad” state in which the firm exerts low effort and
obtains the minimal stage profit $\rho$. The ratings here have no impact on the maximal equilibrium profit and firm’s behaviors.

4 Discussion

The preceding analysis has a number of features that invites further discussion. The rater faces no restriction on the amount of past data to utilize. This assumption is appealing for the question of interest, given the rapidly growing data storage technologies in modern information systems. Yet, it also makes the stark simplicity of the menu $\Xi^*$ rather surprising. The system $S^*_C$ requires one-period memory, and $S^*_I$ is memoryless. The menu, in turn, creates a stationary environment. The stationarity is reflected in the rater’s optimal payoff, which is independent of $\delta$. It is because the short-lived consumers observe nothing but the latest rating. If consumers are infinitely-lived or if the entering short-lived consumers observe all previous ratings, the consumers may learn purely through the qualities or the ratings they observe and allow the competent firm to eventually rest on its laurels. Once an equilibrium calls for the competent type to shirk after some history, the rater’s payoff is strictly less than $\mu(1 - 2\rho - c)$ because it extracts a surplus strictly smaller than $1 - \rho - c$ from the competent type. Nonetheless, it does not overturn the fundamental insight that revenue maximization by the rater calls for maximizing the firm’s expected value to consumers and hence maximizing efficiency, despite the rater’s optimal payoff might now become much harder to keep track of.

In keeping the rater’s payoff tractable, much of the analysis rests upon Assumption 1. It is natural to ask what the rater is capable of achieving when this assumption is violated, while high effort remains efficient. Proposition 5 below shows that when effort cost is sufficiently high, the rater adds no value to the market and extracts no positive surplus. The proof appears in Appendix B.6.

**Proposition 5.** Suppose that $c \in (\bar{c}_{MH}, 1 - 2\rho)$. The competent firm never exerts high effort in any equilibrium, and the rater’s optimal payoff is 0.

The optimal payoff in the intermediate range $c \in (\bar{c}, \bar{c}_{MH}]$ remains an open problem. The bottom line is that the rater can always choose to fully solve the adverse selection problem upon participation by implementing the menu $\Xi_{MH}$. The rating fee $f_{MH}$ is high enough so that in an equilibrium where the competent type frequently exerts
high effort, the inept type chooses not to participate. The rater thus makes the continuation problem one with pure moral hazard, guaranteeing a positive payoff for $c \leq \bar{c}_{MH}$. Specifically:

**Proposition 6.** Suppose that $c \in (0, \bar{c}_{MH}]$. The rater’s optimal payoff is at least

$$W(c) = \mu \left( 1 - 2\rho - c - \frac{\rho c}{1 - 2\rho} \right).$$

The proof is in Appendix B.7. The rater’s optimal payoff $W(c)$ as a function of effort cost $c$ is illustrated below in Figure 4.

![Figure 4: Rater’s Optimal Expected Payoff](image)

It is also interesting to investigate whether Assumption 1 is necessary for the rater to achieve the first-best payoff $\mu(1 - 2\rho - c)$. Supplementary Appendix D presents a numerical example, showing that the answer is negative in general. Analytically characterizing a tight condition for the rater to achieve first best, however, poses a tractability problem. It requires characterizing a menu that, among all menus, maximally relaxes all incentive constraints for high effort in the induced game. Going further, characterizing the rater’s optimal payoff when she can no longer achieve first best in the intermediate range requires keeping track of histories after which the rater would like high effort and those after which the rater would like the firm to shirk in the game induced by each menu, and optimizing over all menus.
Finally, the optimal menu $\Xi^*$ relies on the assumption that the rater can offer distinct schemes to screen the firm’s type. It is perhaps instructive to also examine the rater’s optimal strategy if she is restricted to use only singleton menus, so that screening of the firm’s type is no longer possible. Indeed, in Supplementary Appendix E, I construct a singleton menu $\Xi^{**}$ that gives the rater the maximal payoff $\mu(1 - 2\rho - c)$ by strengthening Assumption 1 to allow for a stronger reputation effect at the prior. The result is surprising in view of Proposition 1, which says that the competent firm must consistently exert high effort and the rating fee must fully extract the participating firm in the rater-preferred equilibrium. For a singleton menu to be optimal, the single rating fee must bind both types’ participation constraints. This requires both types to extract identical expected profits from consumers upon participation, but consistent high effort arises from the competent firm’s incentive to separate itself from the inept type and to secure higher future consumer payments.

The key rests upon the construction of coarse and opaque ratings that make the competent firm indifferent between high and low effort after every history upon participation, giving both types identical expected profits. The indifference condition can be understood as follows. In any equilibrium in which both types participate, the competent firm must extract weakly higher profits than the inept type does, because it can always mimic the inept type by consistently exerting low effort. The optimal rating fee must therefore bind the inept type’s participation constraint. In doing so, the rater aligns her payoff with the inept type’s expected profit extraction from consumers: the higher is the inept firm’s expected profit, the higher fee the rater can charge. Because an inept firm is likely to deliver a bad track record, transparent ratings that give consumers a better idea about the possible quality histories expose the inept type to being statistically identified by the consumers, when the competent type consistently exerts high effort. In this event, consumer payment is relatively low, hurting the inept firm’s profit and therefore the rater’s payoff. The rater therefore obtains her optimal payoff by delivering ratings that are as uninformative as they can be without disrupting incentives for consistent high effort, leaving the competent type indifferent between both effort levels.
Appendices

The Appendices are divided into two parts. Appendix A collects the formal details omitted from the main text. Appendix B presents the proofs.

A Omitted Details

A.1 Fully Revealing Rating System

For a fully revealing rating system $S_{FR} : H_1 \rightarrow \Delta(R_{FR})$ to be well-defined, the cardinality of the rating set $R_{FR}$ has to be sufficiently large. Because the set of quality histories in each period $t$ is countable, and the union of countably many countable sets are countable, one can construct the rating set $R_{FR}$ that is countable by first partitioning the set $H_1$ for each period $t$ into equivalence classes, each of which contains histories $h_1^t$ with identical quality histories, and second labeling each equivalence class with a distinct element in $\mathbb{R}$, letting $R_{FR}^t$ be a set that collects these labels, and finally letting $R_{FR} := \bigcup_{t=0}^{\infty} R_{FR}^t$. The fully revealing rating system $S_{FR}$ can then be a function that maps each history $h_1$, which contains a particular quality history (in addition to the history of ratings), to the label of the equivalence class identified by that quality history in $R_{FR}$ with probability one.

A.2 The Rating Fees $f^*_C$ and $f^*_I$ in the Menu $\Xi^*$

The rating fees are given by

$$f^*_C = \frac{1}{1 - \delta} \left[ \mu(1 - 2\rho) + \frac{\delta(1 - \mu)^2(1 - \rho)(1 - 2\rho)}{1 - \mu(1 - \rho)} - c \right],$$

$$f^*_I = \frac{1}{1 - \delta} \left[ (1 - \delta)(\rho + \mu(1 - 2\rho)) + \frac{\delta\rho(1 - \mu\rho)}{1 - \mu(1 - \rho)} - \rho \right].$$

A.3 Specification of the Menu $\Xi_{MH}$

The menu $\Xi_{MH} = \{\xi_{MH}\}$, where $\xi_{MH} = (f_{MH}, R_{MH}, S_{MH})$, is defined as follows. In the scheme $\xi_{MH}$, the rating set $R_{MH} = \{0, 1\}$ is binary, and the rating system $S_{MH}$ is
defined piecewise depending on the value of $c$ as follows. If

$$c \leq \frac{\delta(1 - 2\rho)^2}{1 + \delta(1 - 2\rho)},$$

then

$$S_{MH}(h^t_1) = \begin{cases} (1 - \beta) \circ \{1\} + \beta \circ \{0\}, & \text{if } y_{t-1} = y \text{ and } r_{t-1} = 1, \\ 1 \circ \{1\}, & \text{otherwise,} \end{cases}$$

where

$$\beta := \frac{c}{\delta((1 - 2\rho)^2 - c(1 - \rho))}. \quad (A.20)$$

In words, the rating system begins with rating 1, announces rating 0 with probability $\beta$ if the previous rating is 1 and a bad quality takes place, and announces rating 1 otherwise. If

$$c > \frac{\delta(1 - 2\rho)^2}{1 + \delta(1 - 2\rho)},$$

then

$$S_{MH}(h^t_1) = \begin{cases} 1 \circ \{1\}, & \text{if } y_{t-1} = \bar{y} \text{ and } r_{t-1} = 1, \\ \kappa \circ \{1\} + (1 - \kappa) \circ \{0\}, & \text{if } y_{t-1} = \bar{y} \text{ and } r_{t-1} = 0, \\ 1 \circ \{0\}, & \text{otherwise,} \end{cases}$$

where

$$\kappa := \frac{\delta(1 - 2\rho)^2 - c(1 - \delta \rho)}{c\delta(1 - \rho)}. \quad (A.21)$$

In words, the rating system announces a rating 1 with probability $\kappa$ if only good qualities take place in the past, and announces a rating 0 otherwise. The fee $f_{MH}$ is

$$f_{MH} = \frac{1}{1 - \delta} \bigg(1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}\bigg).$$

The probabilities $\beta$ and $\kappa$ are well-defined under Assumption 2.
The corresponding rater-preferred equilibrium is \((\sigma_{MH}, \varphi_{MH})\), where \(\sigma_{MH} = (\pi_{MH}, \tau_{MH})\) satisfies

\[
\pi_{MH}(\xi_{MH}|C) = 1, \\
\tau_{MH}(h_2, C) = \begin{cases} 
1, & \text{if } r_t = 1 \text{ and } d = \xi_{MH}, \\
0, & \text{otherwise},
\end{cases}
\]

so that the competent type participates for sure. It chooses high effort with probability one following a rating 1 upon participating in the menu. Otherwise, it exerts low effort with probability one. In a pure moral hazard setting, the belief system \(\varphi_{MH}\) is trivial. Consumers always believe the firm is competent with probability one.

**B Proofs**

This Appendix collects the proofs. Throughout, I write the competent firm’s effort strategy \(\tau(\cdot, C)\) simply as \(\tau(\cdot)\) when there is no risk of ambiguity, because an inept type only exerts low effort. Second, for each \(i = 1, 2\) and two histories \(h_i, h_i' \in H_i\), the notation \(h_i h_i'\) denotes the concatenation of \(h_i\) followed by \(h_i'\), and also belongs to the set \(H_i\).

**B.1 Equilibrium Existence**

**Lemma 1.** For any menu \(\Xi\), the set \(B(\Xi)\) is non-empty.

**Proof.** Consider first an arbitrary menu \(\Xi\) in which the fees are non-negative. In any game induced by \(\Xi\), the profile \((\sigma, \varphi)\) where \(\pi(C) = \pi(I) = 0\) and \(\tau(h_2, C) = \tau(h_2, I) = 0\) for each history \(h_2\), and in which consumers form beliefs \(\varphi_t(r) = 0\) upon seeing any non-null rating \(r\) in any period \(t\), is an equilibrium. On path, consumers always expect low effort from the firm because consumers do not observe past play. Each type thus receives a payoff \(\rho\). By deviating to participate in some scheme, two possibilities arise. Upon a null rating, consumers do not detect the deviation and still pay the firm \(\rho\). Upon observing any non-null rating, consumers believe the firm is inept for sure and pay the firm \(\rho\). Moreover, by participating, the firm pays the rater at least 0 upfront. Its payoff following a deviation is at most \(\rho\).  

\[\blacksquare\]
B.2 Proof of Proposition 1

The proof proceeds via a succession of lemmas. To begin, Lemmas 2 and 3 show that the rater leverages the presence of adverse selection: any optimal menu must induce a game in which each rater-preferred equilibrium features participation with positive probability by both types.

Specifically, Lemma 2 derives an upper bound on the rater’s payoff given any menu and equilibrium in which at most one type participates in a scheme with positive probability in the induced game.

Lemma 3 then shows that the rater obtains a payoff \( W(\Xi^*, (\sigma^*, \varphi^*)) \) strictly higher than this upper bound given the menu \( \Xi^* \) and the equilibrium \( (\sigma^*, \varphi^*) \) in the induced game in which both types participate in the menu with positive probability.

Lemma 4 shows that \( \mu(1 - 2\rho - c) \) is an upper bound on the rater’s problem (5), and that \( W(\Xi^*, (\sigma^*, \varphi^*)) = \mu(1 - 2\rho - c) \). Thus \( \Xi^* \) is optimal, and \( (\sigma^*, \varphi^*) \) is a rater-preferred equilibrium in the induced game.

Finally, Lemma 5 shows that a menu \( \Xi \) and some equilibrium \( (\sigma, \varphi) \) in the induced game give the rater the payoff \( \mu(1 - 2\rho - c) \) if and only if the conditions stated in the proposition hold, completing the proof.

B.2.1 Leveraging the Presence of Adverse Selection

To set the stage, for each menu \( \Xi \) define

\[
B_2(\Xi) := \{ (\sigma, \varphi) \in B(\Xi) : \pi(\Xi|C) > 0, \pi(\Xi|I) > 0 \} \tag{B.22}
\]

as the set of equilibria in which each type participates in at least one scheme of the menu with positive probability. Without putting any additional structure on a menu \( \Xi \), \( B_2(\Xi) \) can possibly be empty. This is the case when, for example, the rating fees \( (f_C, f_I) \) are set to be sufficiently high. Next, define

\[
W_1 := \sup_{\Xi} \sup_{(\sigma, \varphi) \in B(\Xi) \setminus B_2(\Xi)} W(\Xi, (\sigma, \varphi)), \tag{B.23}
\]

\[
W_2 := \sup_{\Xi : B_2(\Xi) \neq \emptyset} \sup_{(\sigma, \varphi) \in B_2(\Xi)} W(\Xi, (\sigma, \varphi)). \tag{B.24}
\]

The payoff \( W_1 \) specifies the value of the rater’s problem, restricting attention to equilibria in which at most one type participates with positive probability. The
payoff $W_2$ is defined analogously, restricting attention to equilibria in which each type participates in some scheme with positive probability.

**Lemma 2.** It holds that

$$W_1 \leq \mu \left(1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}\right).$$  \hspace{1cm} (B.25)

**Proof.** First, fix a menu $\Xi$ and a candidate equilibrium $(\sigma, \varphi) \in B(\Xi)$ with participation strategies $\pi(\Xi | I) > 0$ and $\pi(\Xi | C) = 0$. In this candidate equilibrium, consumers believe that the participating firm is inept for sure, so that $\varphi_t(r) = 0$ and $p_t^C(r) = \rho$ for any rating $r \in \cup_\theta \text{supp} S_\theta(H_2^1)$ and any period $t$. The lowest reservation payoff by the inept firm to not participate is $\rho$. Respecting the inept type’s participation constraint $U(\sigma, I; \xi_\theta) - (1 - \delta)f_\theta \geq U(\sigma, I; N)$, and the fact that $U(\sigma, I; N) \geq \rho$, the rater’s expected payoff satisfies

$$(1 - \delta)(1 - \mu) \sum_\theta \pi(\xi_\theta | I)f_\theta \leq (1 - \mu) \sum_\theta [U(\sigma, I; \xi_\theta) - \rho] = 0.$$

Next, fix a menu $\Xi$ and a candidate equilibrium $(\sigma, \varphi) \in B(\Xi)$ with participation strategies $\pi(\Xi | C) > 0$ and $\pi(\Xi | I) = 0$. In this candidate equilibrium, consumers’ beliefs satisfy $\varphi_t(r) = 1$ for each rating $r \in \cup_\theta \text{supp} S_\theta(H_2^1)$ and each period $t$. Upon participating in a scheme $\xi_\theta$,

$$U(\sigma, C; \xi_\theta) \leq 1 - \rho - c - \frac{\rho c}{1 - 2\rho}. \hspace{1cm} (B.26)$$

To see this, observe first that in a candidate equilibrium where only the competent type participates with positive probability, the expected period-$t$ profit extraction from consumers by the competent firm conditional on participating in a scheme $\xi_\theta$ strictly increases in effort:

$$\mathbb{E}^P[p_t^C(r) - \tau(h_2)c | r_t = r, \theta = C, \xi_\theta] = \mathbb{E}^P[\rho + (1 - 2\rho)\mathbb{E}^P[\tau(h_2) | r_t = r, \theta = C] - \tau(h_2)c | r_t = r, \theta = C, \xi_\theta]$$

$$= \mathbb{E}^P[\rho + (1 - 2\rho)\tau(h_2) - \tau(h_2)c | r_t = r, \theta = C, \xi_\theta]$$

$$= \mathbb{E}^P[\rho + \tau(h_2)(1 - 2\rho - c) | r_t = r, \theta = C, \xi_\theta].$$

After each history $h_2$, the stage profit is strictly increasing in $\tau(h_2)$ because efficiency
of high effort entails that \( c < 1 - 2\rho \). The highest expected profit extraction by the competent firm in this candidate equilibrium is attained by having the competent firm to exert high effort as frequently as possible. The continuation profit of the competent firm after a history \( h^t_2 \), conditional on participating in a scheme \( \xi_\theta \), is given by

\[
V_C^\sigma(h^t_2; \xi_\theta) := (1 - \delta) \mathbb{E}^P \left[ \sum_{s=t}^\infty \delta^s (p^*_\theta(r_s) - c(e_s)) \right]_{h^t_2, \theta = C, \xi_\theta} \\
= (1 - \delta) (p^*_\theta(r_t) - \tau(h^t_2)c) + \delta \mathbb{E}^P[V_C^\sigma(h^t_2, e_t, y_t)|\theta = C, \xi_\theta],
\]

where the expected continuation profit, for each quality \( y_t \in Y \), is

\[
V_C^\sigma(h^t_2, e_t, y_t; \xi_\theta) := \sum_{r_{t+1} \in R} S_\theta(r_{t+1}|h^t_1 y_t) V_C^\sigma(h^t_2, e_t, y_t, r_{t+1}; \xi_\theta).
\]

Exerting high effort after \( h^t_2 \) gives the competent firm a continuation profit

\[
V_C^\sigma(h^t_2; \xi_\theta) = (1 - \delta) (p^*_\theta(r_t) - c) + \delta [(1 - \rho)V_C^\sigma(h^t_2, \bar{e}, \bar{y}; \xi_\theta) + \rho V_C^\sigma(h^t_2, \bar{e}, y; \xi_\theta)].
\]

Shirking, on the other hand, gives a continuation profit

\[
V_C^\sigma(h^t_2; e; \xi_\theta) = (1 - \delta) p^*_\theta(r_t) + \delta [\rho V_C^\sigma(h^t_2, \bar{e}, \bar{y}; \xi_\theta) + (1 - \rho)V_C^\sigma(h^t_2, e, y; \xi_\theta)].
\]

Observe now that in equilibrium the firm’s flow profit after each history \( h^t_2 \) conditional on participating in \( \xi_\theta \) depends only on the history \( h^t_1 \) (which is included in \( h^t_2 \) by definition), because the histories of ratings determine consumers’ payments and the evolution of ratings depends on \( h^t_1 \). The firm essentially faces a Markov decision problem, with the set of states given by the set of the system’s histories \( H_1 \), and so has a Markov best reply. One can then write for each period \( t \), each history \( h^t_2 \) and each effort choice \( e \in \{\bar{e}, \bar{e}\} \) in the above equations that

\[
V_C^\sigma(h^t_1, r_t, \bar{y}; \xi_\theta) := V_C^\sigma(h^t_2, r_t, \bar{e}, \bar{y}; \xi_\theta) = \sum_{r_{t+1} \in R} S_\theta(r_{t+1}|h^t_1 r_t, \bar{y}) V_C^\sigma(h^t_1, r_t, \bar{y}, r_{t+1}; \xi_\theta),
\]

\[
V_C^\sigma(h^t_1, r_t, y; \xi_\theta) := V_C^\sigma(h^t_2, r_t, e, y; \xi_\theta) = \sum_{r_{t+1} \in R} S_\theta(r_{t+1}|h^t_1 r_t, y) V_C^\sigma(h^t_1, r_t, y, r_{t+1}; \xi_\theta).
\]

The incentive constraint for high effort after \( h^t_2 \), namely \( V_C^\sigma(h^t_2; \xi_\theta) - V_C^\sigma(h^t_2; \bar{e}; \xi_\theta) \geq 0 \),
can be simplified to
\[ V^\sigma_C(h^1_t, r_t, \vec{y}; \xi_\theta) - V^\sigma_C(h^1_t, r_t, \bar{y}; \xi_\theta) \geq \frac{(1 - \delta)c}{\delta(1 - 2\rho)}. \] (B.28)

Recursively substituting the constraint (B.28) for each period \( t \), it follows that
\[ U(\sigma, C; \xi_\theta) = V^\sigma_C(h^2_0; \xi_\theta) \leq 1 - \rho - c - \frac{\rho c}{1 - 2\rho}. \] (B.29)

By deviating to not participate, the competent type gets paid \( \rho \) every period, because consumers believe that it is inept with probability one. The competent firm’s participation constraint in \( \xi_\theta \) is therefore \( U(\sigma, C; \xi_\theta) - (1 - \delta)f_\theta \geq \rho \). The rater’s expected payoff therefore satisfies
\[ (1 - \delta)\mu \sum_{\theta \in \Theta} \pi(\xi_\theta|C)f_\theta \leq \mu \sum_{\theta \in \Theta} \pi(\xi_\theta|C)(U(\sigma, C; \xi_\theta) - \rho) \leq \mu \left(1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}\right), \] (B.30)
as desired.

I now turn to the menu \( \Xi^* \). Lemma 3 below shows that \((\sigma^*, \varphi^*)\) is an equilibrium in the game induced by \( \Xi^* \), and that \( W(\Xi^*, (\sigma^*, \varphi^*)) > W_1 \) under Assumption 1.

**Lemma 3.** Under Assumption 1, \((\sigma^*, \varphi^*) \in B_2(\Xi^*)\), and
\[ W(\Xi^*, (\sigma^*, \varphi^*)) = \mu(1 - 2\rho - c) > W_1. \]

**Proof.** The proof is divided into two steps. Step 1 first shows that \((\sigma^*, \varphi^*) \in B_2(\Xi^*)\). Step 2 then derives the value \( W(\Xi^*, (\sigma^*, \varphi^*)) \). This permits a comparison against the value \( W_1 \) derived in Lemma 2 and completes the proof.

**Step 1.** Fix the candidate equilibrium profile \((\sigma^*, \varphi^*)\). Consider first the continuation game after a competent firm chooses participation \( d = \xi^*_C \). Because the rating system \( S^*_C \) specified by the scheme \( \xi^*_C \) relies only on the most recent quality, and there are only two possible ratings \( r \in \{0, 1\} \), and because the firm’s continuation effort strategy after each history is identical conditional on each rating, there are only two possible stage payoffs upon participation by a firm for each period \( t \geq 1 \):
\[ p^*_t(0) - c = \rho + (1 - 2\rho)\varphi^*_t(0) - c = \rho + \frac{(1 - 2\rho)\mu}{\mu + 1 - \mu} - c, \]
\[ p_t^*(1) - c = \rho + (1 - 2\rho)\varphi_t^*(1) - c = 1 - \rho - c. \]

The play by the competent firm from \( t \geq 1 \) upon participation \( d = \xi_C^* \) can be represented by a two-state automaton, with the set of states given by \( R = \{0, 1\} \).

Here, a state \( r \) collects all firm’s histories with the most recent realized rating being \( r \). Let \( V_C^\sigma^*(r; \xi_C^*) \) be the continuation value of the competent firm in a state \( r \) of the automaton given the profile \((\sigma^*, \varphi^*)\) conditional on participating in \( \xi_C^* \). In state \( r \), by exerting high effort, the competent firm’s continuation profit is

\[ V_C^\sigma^*(r; \xi_C^*) = (1 - \delta)(p_t^*(r) - c) + \delta[(1 - \rho)V_C^\sigma^*(1; \xi_C^*) + \rho V_C^\sigma^*(0; \xi_C^*)]. \]

A deviation to shirk yields

\[ V_C^\sigma^*(r; e; \xi_C^*) := (1 - \delta)p_t^*(r) + \delta[\rho V_C^\sigma^*(1; \xi_C^*) + (1 - \rho) V_C^\sigma^*(0; \xi_C^*)]. \]

For \( \tau^* \) to be an equilibrium effort strategy conditional on \( d = \xi_C^* \), the incentive constraint \( V_C^\sigma^*(r; \xi_C^*) - V_C^\sigma^*(r; e; \xi_C^*) \geq 0 \) in a state \( r \) needs to hold, for each \( r \in \{0, 1\} \).

The constraint can be simplified as

\[ \delta(1 - 2\rho)(p_t^*(1) - p_t^*(0)) \geq c, \quad (B.31) \]

or equivalently

\[ \delta(1 - 2\rho)^2 \left[1 - \frac{\mu\rho}{1 - \mu + \mu\rho}\right] \geq c. \]

Because the left hand side equals \( \bar{c}(\mu, \delta, \rho) \), the incentive constraint holds by Assumption 1. In period 0, each type of firm receives a payment of \( \rho + (1 - 2\rho)\mu \) upon participation, and the competent firm faces the same incentive constraint \((B.31)\), which therefore is also satisfied.

Consider now the competent firm’s behavior if it has chosen \( d = \xi_I^* \). Because only rating 0 is announced, the competent firm finds it optimal to always exert low effort. Formally, the play by the competent firm from \( t \geq 0 \) upon participation \( d = \xi_I^* \) can be represented by a one-state automaton, with the state denoted by \( r = 0 \) which collect all the firm’s histories with the most recent realized rating being 0. Let \( V_C^\sigma^*(r; \xi_I^*) \) be the continuation value of the competent firm in a state \( r \) of the automaton given the
profile \((\sigma^*, \varphi^*)\) conditional on participating in \(\xi^I_r\). In state \(r = 0\), by exerting high effort, the competent firm’s continuation profit is

\[
V^{\sigma^*}_C(0; \xi^C) = (1 - \delta)(p^0_{\sigma^*}(0) - c) + \delta V^{\sigma^*}_C(0; \xi^C).
\]

A deviation to shirk yields

\[
V^{\sigma^*}_C(0; e; \xi^C) := (1 - \delta)p^0_{\sigma^*}(0) + \delta V^{\sigma^*}_C(0; \xi^C).
\]

The claim thus follows because the incentive constraint is always violated, namely \(V^{\sigma^*}_C(0; \xi^C) - V^{\sigma^*}_C(0; e; \xi^C) < 0\).

Similarly, suppose now the competent firm chooses \(d = N\). Consumers believe that the firm is inept for sure, and therefore pays the firm \(\rho\) every period. The competent firm finds it optimal to exert low effort in every period in the continuation game.

It remains to check that the specified participation decisions by \(\pi^*\) are indeed equilibrium phenomena. Upon participating in \(d = \xi^C\) and paying the fee \(f^*_C\), the competent firm’s payoff equals

\[
U(\sigma^*, C; \xi^C) - (1 - \delta)f^*_C
= (1 - \delta)(p^0_{\sigma^*}(0) - c) + (1 - \delta)\sum_{t=1}^{\infty} \delta^t[(1 - \rho)(p^t_{\sigma^*}(1) - c) + \rho(p^t_{\sigma^*}(0) - c)] - (1 - \delta)f^*_C
= \rho + \mu(1 - 2\rho) + \frac{\delta(1 - \mu)^2(1 - \rho)(1 - 2\rho)}{1 - \mu(1 - \rho)} - c - (1 - \delta)f^*_C
= \rho + \mu(1 - 2\rho) + \frac{\delta(1 - \mu)^2(1 - \rho)(1 - 2\rho)}{1 - \mu(1 - \rho)} - c - \left[\mu(1 - 2\rho) + \frac{\delta(1 - \mu)^2(1 - \rho)(1 - 2\rho)}{1 - \mu(1 - \rho)} - c\right]
= \rho.
\]

By deviating to choose \(d = \xi^I\) and paying the fee \(f^*_I\), the competent firm’s payoff equals

\[
U(\sigma^*, C; \xi^I) - (1 - \delta)f^*_I
= (1 - \delta)p^0_{\sigma^*}(0) + (1 - \delta)\sum_{t=1}^{\infty} \delta^tp^t_{\sigma^*}(0) - (1 - \delta)f^*_I
= (1 - \delta)(\rho + \mu(1 - 2\rho)) - \frac{\delta\rho(1 - \mu\rho)}{1 - \mu(1 - \rho)} - (1 - \delta)f^*_I
\]
\[(1 - \delta)(\rho + \mu(1 - 2\rho)) - \frac{\delta \rho (1 - \mu \rho)}{1 - \mu (1 - \rho)} - \left[ (1 - \delta)(\rho + \mu(1 - 2\rho)) + \frac{\delta \rho (1 - \mu \rho)}{1 - \mu (1 - \rho)} - \rho \right] = \rho.\]

By deviating to choose \(d = N\), the competent firm gets paid \(\rho\) every period, and its payoff therefore equals

\[U(\sigma^*, C; N) - 0 = \rho.\]

Thus, the competent type does not find a profitable deviation from \(\xi^*_C\) to either \(\xi^*_I\) or \(N\). Finally, for the inept type, upon participating in \(d = \xi^*_I\) and paying the fee \(f^*_I\), its payoff equals

\[U(\sigma^*, I, \xi^*_I) - (1 - \delta)f^*_I = (1 - \delta)p^*_0(0) + (1 - \delta)\sum_{t=1}^{\infty} \delta^t p^*_t(0) - (1 - \delta)f^*_I = (1 - \delta)(\rho + \mu(1 - 2\rho)) - \frac{\delta \rho (1 - \mu \rho)}{1 - \mu (1 - \rho)} - (1 - \delta)f^*_I = (1 - \delta)(\rho + \mu(1 - 2\rho)) - \frac{\delta \rho (1 - \mu \rho)}{1 - \mu (1 - \rho)} - \left[ (1 - \delta)(\rho + \mu(1 - 2\rho)) + \frac{\delta \rho (1 - \mu \rho)}{1 - \mu (1 - \rho)} - \rho \right] = \rho.\]

By deviating to \(d = \xi^*_C\) and paying the fee \(f^*_C\), its payoff equals

\[U(\sigma^*, I, \xi^*_C) - (1 - \delta)f^*_C = (1 - \delta)p^*_0(0) + (1 - \delta)\sum_{t=1}^{\infty} \delta^t [\rho(p^*_t(1)) + (1 - \rho)(p^*_t(0))] - (1 - \delta)f^*_C = \frac{c - \delta(1 - \mu)(1 - 2\rho)^2 - c \mu (1 - \rho)}{1 - \mu(1 - \rho)} \leq \rho\]

whenever \(c \leq \tilde{c}(\delta, \mu, \rho)\). Finally, by deviating to choose \(d = N\), the inept firm gets paid \(\rho\) every period, and its payoff equals

\[U(\sigma^*, I; N) - 0 = \rho.\]
Therefore, the inept type does not find a profitable deviation from $\xi_i^*$ to either $\xi_C^*$ or $N$. Thus $\pi^*$ is indeed an equilibrium participation strategy, and as a result, $(\sigma^*, \varphi^*) \in B_2(\Xi^*)$.

**Step 2.** The rater’s payoff using the menu $\Xi^*$ in the equilibrium $(\sigma^*, \varphi^*)$ is

$$W(\Xi^*, (\sigma^*, \varphi^*)) = (1 - \delta)[\mu f_C^* + (1 - \mu) f_I^*]$$

$$= \mu (1 - 2\rho - c) > \mu \left(1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}\right) = W_1,$$

as desired.  

**B.2.2 Upper Bound on the Rater’s Payoff**

The previous results immediately imply that

$$W_2 \geq W(\Xi^*, (\sigma^*, \varphi^*)) > W_1,$$

where the first inequality follows by definition. Thus, any optimal menu must induce a game in which each rater-preferred equilibrium features participation with positive probability by both types. The rater’s problem can therefore be reformulated as

$$\sup_{\Xi} \sup_{(\sigma, \varphi) \in B_2(\Xi)} W(\Xi, (\sigma, \varphi))$$

subject to

- $U(\sigma, C; \xi_C) - (1 - \delta) f_C \geq U(\sigma, C; N)$, \hspace{1cm} (IR$_C$)
- $U(\sigma, I; \xi_I) - (1 - \delta) f_I \geq U(\sigma, I; N)$, \hspace{1cm} (IR$_I$)
- $U(\sigma, C; \xi_C) - (1 - \delta) f_C \geq U(\sigma, C; \xi_I) - (1 - \delta) f_I$, \hspace{1cm} (IC$_C$)
- $U(\sigma, I; \xi_I) - (1 - \delta) f_I \geq U(\sigma, I; \xi_C) - (1 - \delta) f_C$, \hspace{1cm} (IC$_I$)

Lemma 4 below establishes an upper bound on this rater’s problem, showing that the bound equals $\mu(1 - 2\rho - c)$. Because $W(\Xi^*, (\sigma^*, \varphi^*)) = \mu(1 - 2\rho - c)$, the proof of Proposition 1 is complete. In the proof, for each firm’s history $h^2_t$, write $h^2_{t,1} \in H^1_t$ as the history of qualities and ratings (i.e. the firm’s track record) embedded in $h^2_t$ that are taken as input in the rating systems $S_C$ and $S_I$. 

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Lemma 4. Given any feasible candidate solution \((\Xi,(\sigma,\varphi))\) of the above problem,

\[ W(\Xi,(\sigma,\varphi)) \leq \mu(1 - 2\rho - c). \]

Proof. The rater’s payoff given any feasible candidate solution \((\Xi,(\sigma,\varphi))\) of the above problem must satisfy

\[
W(\Xi,(\sigma,\varphi)) = (1 - \delta)E^\mu \left[ \sum_{\varphi'} \pi(\xi_{\varphi'}|\theta) f_{\varphi'} \right]
\]

\[
\leq E^\mu \left[ \sum_{\varphi'} \pi(\xi_{\varphi'}|\theta)(U(\sigma,\theta;\xi_{\varphi'}) - U(\sigma,\theta;N)) \right]
\]

\[
\leq E^\mu \left[ \sum_{\varphi'} \pi(\xi_{\varphi'}|\theta)(U(\sigma,\theta;\xi_{\varphi'}) - \rho) \right], \tag{B.32}
\]

where the inequality in (B.32) follows from the IR and IC constraints: when \(\theta = \theta'\), the IR constraint for each type \(\theta\) implies that \((1 - \delta)f_\theta \leq U(\sigma,\theta;\xi_\theta) - U(\sigma,\theta;N) \leq U(\sigma,\theta;\xi_\theta) - \rho\). The second inequality follows because \(\rho\) is the lowest possible payoff. When \(\theta \neq \theta'\) and if \(\pi(\xi_{\varphi'}|\theta) > 0\) in equilibrium, then it must be true that the IC constraint for type \(\theta\) binds. By the IR constraint for type \(\theta\) then, \((1 - \delta)f_{\varphi'} \leq U(\sigma,\theta;\xi_{\varphi'}) - U(\sigma,\theta;N) \leq U(\sigma,\theta;\xi_{\varphi'}) - \rho\) and therefore the inequality in (B.32) follows. Continuing from (B.32),

\[
W(\Xi,(\sigma,\varphi)) \leq (1 - \delta) \sum_{t=0}^{\infty} \delta^t \left\{ E^\mu \left[ \sum_{\varphi'} \pi(\xi_{\varphi'}|\theta) \sum_{h_{2,1}^t} P_t^\theta(h_{2,1}^t|\xi_{\varphi'}) \sum_{r_t} S_{\varphi'}(r_t|h_{2,1}^t) p^\varphi_t(r_t) \right] - \rho \right\}
\]

\[
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \left\{ E^\mu \left[ \sum_{\varphi'} \pi(\xi_{\varphi'}|\theta) \sum_{h_{2,1}^t} P_t^\theta(h_{2,1}^t|\xi_{\varphi'}) \sum_{r_t} S_{\varphi'}(r_t|h_{2,1}^t) \right. \right.
\]

\[
\times \left. (\rho + (1 - 2\rho)\varphi_t(r_t)) E^\mu[\tau(h_{2}^t)|r_t,\theta = C] - \rho \right) \right\}
\]

\[
- \mu \sum_{\varphi'} \pi(\xi_{\varphi'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\varphi'}) \sum_{r_t} S_{\varphi'}(r_t|h_{2,1}^t) \tau(h_{2}^t,\xi_{\varphi'})c \right\} \tag{B.33}
\]
\[
(1 - \delta) \sum_{t=0}^{\infty} \delta^t \left\{ \mathbb{E}^\mu \left[ \sum_{\theta'} \pi(\xi_{\theta'}|\theta) \sum_{h_{2,1}^t} P_t^0(h_{2,1}^t|\xi_{\theta'}) \right. \\
\left. \times \left( 1 - 2\rho \right) \sum_{r_t} S_{\theta'}(r_t|h_{2,1}^t) \right] \right\} - \mu \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) \sum_{r_t} S_{\theta'}(r_t|h_{2,1}^t) \tau(h_{2,1}^t, \xi_{\theta'}) c \right\}
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \left\{ (1 - 2\rho) \sum_{r_t} \mathbb{E}^\mu \left[ \sum_{\theta'} \pi(\xi_{\theta'}|\theta) \sum_{h_{2,1}^t} P_t^0(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \right] \times \\
\frac{\mu \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t)}{\mathbb{E}^\mu \left[ \sum_{\theta'} \pi(\xi_{\theta'}|\theta) \sum_{h_{2,1}^t} P_t^0(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \right]} \mathbb{E}^P[\tau(h_{2,1}^t)|r_t, \theta = C] \\
\times \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \tau(h_{2,1}^t, \xi_{\theta'}) c \right\}
\]
(\text{B.34})

\[
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{r_t} \left\{ (1 - 2\rho) \mu \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \times \\
\sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \tau(h_{2,1}^t, \xi_{\theta'}) \right\} \\
= (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{r_t} \left\{ (1 - 2\rho) \mu \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \tau(h_{2,1}^t, \xi_{\theta'}) \\
- \mu c \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) S_{\theta'}(r_t|h_{2,1}^t) \tau(h_{2,1}^t, \xi_{\theta'}) \right\} \right\}
= \mu(1 - 2\rho - c) (1 - \delta) \sum_{t=0}^{\infty} \delta^t \sum_{r_t} \pi(\xi_{\theta'}|C) \sum_{h_{2,1}^t} P_t^C(h_{2,1}^t|\xi_{\theta'}) \sum_{r_t} S_{\theta'}(r_t|h_{2,1}^t) \tau(h_{2,1}^t, \xi_{\theta'}) \leq 1
\]
\[
\leq \mu(1 - 2\rho - c)(1 - \delta)\sum_{t=0}^{\infty} \delta^t \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{h_2^t} P^C(h_2^t|\xi_{\theta'}) \sum_{r_t} S_{\theta'}(r_t|h_2^t,1) \tag{B.39}
\]

\[
= \mu(1 - 2\rho - c) \sum_{\theta'} \pi(\xi_{\theta'}|C) \sum_{r_t} S_{\theta'}(r_t|h_2^t,1) \leq 1 \tag{\leq 1}
\]

\[
\leq \mu(1 - 2\rho - c), \tag{B.40}
\]

as desired. In the above derivation,

- (B.33) follows by substituting \( p^\sigma_t(r) \) using (6),
- (B.34) follows by substituting \( \varphi_t(r_t) \) using (1),
- (B.35) rearranges the order of summations,
- (B.36) expresses the expected effort explicitly according to the corresponding conditional distribution,
- (B.37) follows from simplifying (B.36),
- (B.38) follows from factorizing out the term \( \mu(1 - 2\rho - c) \),
- (B.39) follows because the effort probability is bounded above by one,
- and (B.40) follows as the participation probability is bounded above by one.

\[\square\]

### B.2.3 Attaining the Upper Bound

**Lemma 5.** For every menu \( \Xi \) and equilibrium \( (\sigma, \varphi) \in B(\Xi) \), \( W(\Xi, (\sigma, \varphi)) = \mu(1 - 2\rho - c) \) if and only if the conditions stated in Proposition 1 are satisfied.

**Proof.** Because the rater obtains the payoff \( \mu(1 - 2\rho - c) \) using the menu \( \Xi^* \) in the equilibrium \( (\sigma^*, \varphi^*) \), any menu \( \Xi \) that is optimal must also give the rater the same payoff in some equilibrium in the corresponding induced game. Now, fix an optimal menu \( \Xi \), and suppose that there is no equilibrium \( (\sigma, \varphi) \in B(\Xi) \) that satisfies all of the stated conditions in the proposition. Fix an arbitrary equilibrium \( (\sigma, \varphi) \in B(\Xi) \). First, if \( (\sigma, \varphi) \notin B_2(\Xi) \), the menu \( \Xi \) cannot be optimal by Lemma 3. Second, if \( (\sigma, \varphi) \in B_2(\Xi) \) but at least one of the stated equilibrium properties does not hold, then at least one of the inequalities in the derivation in Lemma 4 becomes strict, giving
Because $(\sigma, \varphi) \in B_2(\Xi)$ is arbitrarily chosen, $\Xi$ is not optimal. Conversely, fix a menu $\Xi$ and an equilibrium $(\sigma, \varphi) \in B(\Xi)$ that satisfy the conditions. Then all the inequalities in the derivation in Lemma 4 become equalities, giving $W(\Xi, (\sigma, \varphi)) = \mu(1 - 2\rho - c)$, so that $\Xi$ is optimal.

**B.3 Proof of Proposition 2**

For any menu $\Xi$ and any equilibrium $(\sigma, \varphi) \in B_2(\Xi)$, it follows from a derivation analogous to that in Lemma 4 that $W(\Xi, (\sigma, \varphi)) \leq \mu(1 - 2\rho)$, with the restrictions $c = 0$ and $\tau(\cdot, C) = 1$. The menu $\Xi^*$ and the equilibrium $(\sigma^*, \varphi^*)$ with these restrictions give $W(\Xi^*, (\sigma^*, \varphi^*)) = \mu(1 - 2\rho)$, so that a menu $\Xi$ is optimal if and only if there exists an equilibrium $(\sigma, \varphi)$ in the induced game that gives the rater $W(\Xi, (\sigma, \varphi)) = \mu(1 - 2\rho)$. The first set of conditions in the proposition is a direct counterpart of Proposition 1.

For the second set of conditions, for any equilibrium $(\sigma, \varphi) \in B(\Xi) \setminus B_2(\Xi)$ the rater extracts her highest possible payoff when only the competent type participates and it participates with probability one, because high effort is efficient. In such an equilibrium, the highest rating fee $f$ a rater can charge the one that binds the competent firm’s participation constraint in any scheme $\xi$ it participates: $f = (U(\sigma, C; \xi) - \rho)/(1 - \delta)$. Here, because the competent firm’s type is perfectly revealed upon participation, $U(\sigma, C; \xi) = 1 - \rho$ given any rating system $S$. The rater expects to collect from each scheme the competent firm participates with positive probability an expected payoff of $\mu(1 - 2\rho)$. The rater obtains this payoff in an equilibrium in which at most one type participates with positive probability if and only if the stated conditions hold. The set of menus and equilibria in the induced game satisfying these conditions is non-empty, containing at least $\Xi'_AS$ and the equilibrium in the induced game described in the main text.

**B.4 Proof of Proposition 3**

For any scheme $\xi$ and any equilibrium strategy profile $\sigma$, it follows from an analogous argument as in Lemma 2 that

$$U(\sigma, C; \xi) \leq 1 - \rho - c - \frac{\rho c}{1 - 2\rho}.$$
Respecting individual rationality, the firm must obtain at least a payoff \( \rho \) by participating (because \( \rho \) is the lower bound on its payoff by choosing the outside option). The rater’s payoff using any menu \( \Xi \) in any equilibrium \((\sigma, \varphi)\) is thus at most

\[
1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}.
\]

It remains to show that \((\sigma_{MH}, \varphi_{MH})\) is indeed an equilibrium in the game induced by the menu \(\Xi_{MH} = \{\xi_{MH}\}\), and that the menu gives the rater a payoff equal to this upper bound in the equilibrium. Consider first the case

\[
c \leq \frac{\delta(1 - 2\rho)^2}{1 + \delta(1 - 2\rho)}.
\]

I show that \((\sigma_{MH}, \varphi_{MH}) \in B(\Xi_{MH})\). Given \((\sigma_{MH}, \varphi_{MH})\), after any history of play from period \(t \geq 1\), upon each rating \(r \in \{0, 1\}\), the firm’s stage profit and its continuation strategy by the firm are identical. Consequently the play upon participation by the competent firm can be represented by a two-state automaton, with states given by the ratings \(r \in \{0, 1\}\) and transitions depending on the current qualities, depicted by Figure 5 below. The initial state is 1, in which the firm chooses high effort with probability 1; in state 0, the firm chooses low effort with probability 1.

![Figure 5: The two-state automation](image)

Denote the competent firm’s continuation profit extraction from consumers in a state \(r \in \{0, 1\}\) by \(V_{r}^{\sigma_{MH}}\). The continuation profits upon participation are:

\[
V_{1}^{\sigma_{MH}} = (1 - \delta)(1 - \rho - c) + \delta((1 - \rho \beta)V_{1}^{\sigma_{MH}} + \rho \beta V_{0}^{\sigma_{MH}}),
\]

\[
V_{0}^{\sigma_{MH}} = (1 - \delta)\rho + \delta V_{1}^{\sigma_{MH}}.
\]

Solving for \(V_{1}^{\sigma_{MH}}\) and \(V_{0}^{\sigma_{MH}}\), it follows that the incentive constraint for high effort in state 1 is satisfied, because \(\delta(1 - 2\rho)\beta(V_{1}^{\sigma_{MH}} - V_{0}^{\sigma_{MH}}) \geq (1 - \delta)c\) under Assumption 2.
Clearly, the incentive constraint for high effort in state 0 is violated, so that the firm finds it optimal to exert low effort. Straightforward algebraic manipulation shows

\[ V_{1}^{\sigma_{MH}} = 1 - \rho - c - \frac{\rho c}{1 - 2\rho}. \]

Next, by deviating to not participate, the firm finds it optimal to always exert low effort, because future consumers do not observe past qualities. Each entering consumer thus pays the firm exactly \( \rho \) upon observing a null rating, giving \( U(\sigma_{MH}, C; N) = \rho \).

It remains to verify that the competent firm does not deviate from participating. This follows because the competent firm who participates obtains

\[
U(\sigma_{MH}, C; \xi_{MH}) - (1 - \delta)f_{MH}
= 1 - \rho - c - \frac{\rho c}{1 - 2\rho} - \left(1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}\right) = \rho = U(\sigma_{MH}, C; N).
\]

Consider next the case

\[ c \in \left(\frac{\delta(1 - 2\rho)^2}{1 + \delta(1 - 2\rho)}, \bar{c}_{MH}\right). \]

An analogous two-state automaton can be constructed in Figure 6 below. The initial state is 1, in which the firm chooses high effort with probability 1; in state 0, the firm chooses low effort with probability 1.

![Figure 6: The two-state automaton](image)

The continuation profits \( V_{1}^{\sigma_{MH}}, V_{0}^{\sigma_{MH}} \) satisfy:

\[ V_{1}^{\sigma_{MH}} = (1 - \delta)(1 - \rho - c) + \delta((1 - \rho)V_{1}^{\sigma_{MH}} + \rho V_{0}^{\sigma_{MH}}), \]
\[ V_{0}^{\sigma_{MH}} = (1 - \delta)\rho + \delta((1 - \rho)\kappa V_{1}^{\sigma_{MH}} + (1 - (1 - \rho)\kappa)V_{0}^{\sigma_{MH}}). \]

Again, the incentive constraint for high effort in state 1 is satisfied, because \( \delta(1 - \)
2\rho)(V_1^{\sigma MH} - V_0^{\sigma MH}) \geq (1 - \delta)c \text{ by construction under Assumption 2.} \quad \text{Also, the incentive constraint for high effort in state 0 is violated, so that the firm finds it optimal to exert low effort. Straightforward algebraic manipulation again reveals that}

\[
V_1^{\sigma MH} = 1 - \rho - c - \frac{\rho c}{1 - 2\rho}.
\]

It is then analogous to the first case to show that the competent firm does not profit from deviating to not participate.

**B.5 Proof of Proposition 4**

The proof is omitted because it is identical to showing (B.26) in Lemma 2.

**B.6 Proof of Proposition 5**

In the setting with both adverse selection and moral hazard, the firm’s incentive constraint for high effort after each history \(h_2\) is

\[
\delta(1 - 2\rho)(V(h_2\bar{y}) - V(h_2y)) \geq c,
\]

where \(V(h'_2)\) denotes the continuation profit of the competent firm after any history \(h'_2\). By an analogous argument that proves (B.26) in Lemma 2,

\[
V(h_2\bar{y}) \leq 1 - \rho - c - \frac{\rho c}{1 - 2\rho}.
\]

Further, \(V(h_2y) \geq \rho\), because the firm can guarantee a profit of \(\rho\) by consistently exerting low effort. The incentive constraint therefore implies that

\[
\delta(1 - 2\rho)\left(1 - 2\rho - c - \frac{\rho c}{1 - 2\rho}\right) \geq c,
\]

must hold. Because the left side strictly decreases in \(c\) and the right side strictly increases in \(c\) and they are equal at \(c = \bar{c}_{MH}\), the incentive constraint for high effort must be violated whenever \(c > \bar{c}_{MH}\). In any equilibrium, the firm’s profit equals \(\rho\). Respecting individual rationality of participating in a scheme, the rater must compensate a participating firm with at least \(\rho\) and thus obtain a payoff 0.

**B.7 Proof of Proposition 6**

Fix the menu \(\Xi_{MH}\) and an equilibrium in which only the competent participates in the
menu with positive probability. In addition, it participates with probability one. The competent firm’s effort strategy follows $\tau_{MH}$. If it chooses the outside option, then it always exerts low effort. By Lemma 2, the rater’s expected payoff in an equilibrium in which only the competent firm participates with positive probability is bounded above by (19). It is analogous to the proof of Lemma 5 to show that the prescribed profile constitutes an equilibrium. In addition, the rater’s expected payoff in this equilibrium given the menu $\Xi_{MH}$ is precisely (19), completing the proof.

References


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Supplementary Appendices to “The Dual Role of Ratings”

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January 27, 2019

The following appendices contain material omitted from the main text. The proofs of the propositions in these appendices appear in Section F.

C Perfect Monitoring

This section characterizes optimal menus when $\rho = 0$, i.e. when there is perfect monitoring of the firm’s effort choice by the rater. A good quality perfectly reveals that the firm has exerted high effort, and a bad quality perfectly reveals that the firm has exerted low effort. The assumption below simply rewrites Assumption 1 with $\rho = 0$.

Assumption 3 (Effective Intermediation with Perfect Monitoring). The parameters $\delta, c$ satisfy

$$c \leq \delta. \quad \text{(C.42)}$$

Proposition 7 (Necessary and Sufficient Conditions for Optimal Menus with Perfect Monitoring). Under Assumption 3, a menu $\Xi$ is optimal if and only if one of the following conditions is true:

A. there exists an equilibrium $(\sigma, \varphi) \in B(\Xi)$ such that

1. $\pi(\Xi|C) = 1$,
2. $\pi(\Xi|I) > 0$,
3. $\tau(h^2_{t}, C) = 1$ after every history $h^2_{t}$ that occurs with positive probability for each $t \geq 0$ conditional on the firm’s type being competent,
4. for each type $\theta$, if $\pi(\xi_{\theta^}\theta) > 0$, then $(1 - \delta)\xi_{\theta^} = U(\sigma, \theta; \xi_{\theta^}) - \rho$.

B. there exists an equilibrium $(\sigma, \varphi) \in B(\Xi)$ such that

1. $\pi(\Xi|C) = 1$,
2. $\pi(\Xi|I) = 0$,
3. $\tau(h^2_{t}, C) = 1$ after every history $h^2_{t}$ that occurs with positive probability for each $t \geq 0$ conditional on the firm’s type being competent,
4. If \( \pi(\xi^\prime|C) > 0 \), then \((1 - \delta)f_{\theta} = U(\sigma, C, \xi^\prime) - \rho\).

The rater’s optimal payoff is \( \mu(1 - c) \).

The conditions in Part A are familiar from Proposition 1. The set of optimal menus expand relative to Proposition 1 because it is now possible to induce consistent high effort in an equilibrium in which only the competent type participates, contrary to the imperfect monitoring case. Specifically, there is a singleton menu with an equilibrium in the induced game satisfying the conditions in Part B. The corresponding rating system assigns “bad” ratings inducing low effort forever whenever one bad quality is observed, and a “good” rating otherwise. Under Assumption 3, the participating competent firm consistently exerts high effort to avoid the perpetual punishment of a low payment associated with low effort.

D Further Discussion on Assumption 1

This section presents a numerical example showing that it is not necessary for first best efficiency in general. Let the discount factor \( \delta = 0.9 \), the monitoring error \( \rho = 0.1 \) and the prior \( \mu = 0.8 \). The threshold in Assumption 1 is

\[
\bar{c} = \delta(1 - 2\rho)^2 \left( \frac{1 - \mu}{1 - \mu + \mu \rho} \right) = 0.411429.
\]

The threshold above which the rater obtains zero payoff is

\[
\bar{c}_{MH} = \frac{\delta(1 - 2\rho)^2}{1 - \delta \rho} = 0.632967.
\]

Fix \( c = 0.412 \), so that \( \bar{c} < c < \bar{c}_{MH} \). By the preceding argument in Appendix D, the maximal surplus cannot be generated if the rater can use at most two ratings in a scheme. Consider, however, the following menu of rating schemes, denoted \( \Xi = \{\xi_C, \xi_I\} \), and a candidate equilibrium profile \((\sigma, \varphi)\) in the game induced by \( \Xi \). In the “competent” scheme \( \xi_C \), the rating set is \{0, 1, 2\}. The rating system \( S_C \) is depicted in Figure 7 below, with the initial distribution over the rating set being \((0.05, 0.86, 0.09)\), which is the stationary distribution induced by \( \sigma \).

Finally, the rating fee \( f_C \approx 3.13 \) is chosen so that \( U(\sigma; C, \xi_C) - (1 - \delta)f_C = \rho \). In the “inept” scheme, the rating set is \{0, 2\}. The rating system \( S_I \) is depicted in Figure 8 below, with the initial distribution over the rating set being \((0.72, 0.28)\). Finally, the rating fee \( f_I \approx 2.89 \) is chosen so that \( U(\sigma; I, \xi_I) - (1 - \delta)f_I = \rho \).

Let \( \alpha_{rs} \) denote the transition probability from rating \( r \) to rating \( s \) upon a good realized
Figure 7: Transitions in the “Competent” Scheme

The transition probabilities are written in the form $(x, y)$ and are rounded up to two decimal places, where $x$ denotes the transition upon a good quality and $y$ denotes the transition upon a bad quality.

Figure 8: Transitions in the “Inept” Scheme

The transition probabilities are written in the form of $(x, y)$, where $x$ denotes the transition upon a good quality and $y$ the transition upon a bad quality.

quality in the competent scheme, and let $\beta_{rs}$ denote the counterpart upon a bad quality. Their precise values are given in Figure 7 above. The continuation profit $V_{r}^\sigma$ upon each rating $r = 0, 1, 2$ by the competent firm in the candidate equilibrium $(\sigma, \varphi)$ can be expressed in the following Bellman equation:

$$V_{r}^\sigma = (1 - \delta)(p^\sigma(r) - c) + \delta \sum_{s=0}^{2} ((1 - \rho)\alpha_{rs} + \rho \beta_{rs})V_{s}^\sigma.$$

The incentive constraints for high effort by the competent firm upon rating $r$ is therefore

$$\delta(1 - 2\rho)\left[(\alpha_{r0} - \beta_{r0})(V_0 - V_2) + (\alpha_{r1} - \beta_{r1})(V_1 - V_2)\right] \geq (1 - \delta)c.$$
Solving the system of Bellman equations,

\[ V_1^\sigma = 0.43 > V_2^\sigma = 0.4 > V_0^\sigma = 0.34, \]

and using the `NMaximize` function in Mathematica,\(^{18}\) it follows that

\[
\min_{r \in \{0, 1, 2\}} \delta(1 - 2\rho) \left[ (\alpha r_0 - \beta r_0)(V_0^\sigma - V_2^\sigma) + (\alpha r_1 - \beta r_1)(V_1^\sigma - V_2^\sigma) \right] \approx 0.0435,
\]

while \((1 - \delta)c = 0.0412.\(^{19}\) All three incentive constraints are therefore satisfied, and the competent firm consistently exerts high effort upon participation. It is then analogous to the proof of Proposition 1 to verify that neither type finds it strictly profitable to deviate from their participation strategies specified by \(\pi\). Consequently, \((\sigma, \varphi)\) is an equilibrium in the induced game. Because the competent firm consistently exerts high effort, the menu \(\Xi\) generates the maximal expected surplus in the market in the equilibrium. Moreover, the four conditions in Proposition 1 are satisfied by the equilibrium \((\sigma, \varphi)\) and the menu \(\Xi\), implying that the rater obtains \(\mu(1 - 2\rho - c)\) in \((\sigma, \varphi)\) given \(\Xi\).

## E Singleton Optimal Rating Menu

In this section, I impose the following assumption.

**Assumption 4.** The parameters \(\mu, \delta, \rho, c\) satisfy

\[
c \leq \frac{\delta(1 - \mu)(1 - 2\rho)^3}{(1 - \rho - \mu(1 - 2\rho))(\rho + \mu(1 - 2\rho))} =: \bar{c}(\mu, \delta, \rho).
\]  

(E.43)

It is stronger than Assumption 1 in the main text: \(\bar{c}(\mu, \delta, \rho) < \bar{c}(\mu, \delta, \rho)\). To see this, note that the inequality can be simplified to

\[
\frac{1 - \rho}{1 - 2\rho} > (2 - \mu)\mu.
\]

The right side is maximized at \(\mu = 1\), and is therefore bounded above by 1. The left side is strictly larger than 1 for any \(\rho \in (0, \frac{1}{2})\). The difference between the two cost thresholds dissipates as the discount factor \(\delta\) approaches 1 and the monitoring error \(\rho\) approaches 0.

The singleton menu \(\Xi^{**} = \{\xi^{**}\}\), where the scheme \(\xi^{**} = (f^{**}, R^{**}, S^{**})\), is defined as follows. The set of ratings \(R^{**} = \{0, 1\}\) is binary, and the rating system delivers a rating 1

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\(^{18}\)See `numerical3states.nb` for the numerical computation.

\(^{19}\)Because the numerical solution given by the `NMaximize` function can possibly be a local maximum, the global maximum is at least 0.0435.
with probability \( \alpha \) if the most recent quality is good, and delivers a rating 0 otherwise:

\[
S^{**}(h^1) = \begin{cases} 
\alpha \circ \{1\} + (1 - \alpha) \circ \{0\}, & \text{if } y_{t-1} = \bar{y}, \\
1 \circ \{0\}, & \text{otherwise},
\end{cases}
\]  

(E.44)

for each history \( h^1 \), where

\[
\alpha := \frac{c(\rho + \mu(1 - 2\rho))}{c(\rho + \mu(1 - 2\rho))^2 + \delta(1 - \mu)\mu(1 - 2\rho)^3}.
\]  

(E.45)

Assumption 4 ensures that \( \alpha \in (0, 1] \) is a well-defined probability. The fee

\[
f^{**} = \frac{1 - 2\rho}{1 - \delta} \left[ \mu - \frac{\delta\alpha(1 - \mu)\mu^2(1 - 2\rho)^2}{(\rho + \mu(1 - 2\rho))(1 - \alpha(\rho + \mu(1 - 2\rho)))} \right],
\]  

(E.46)

is set to maximally extract both types who participate in the scheme \( \xi^{**} \) in the equilibrium \((\sigma^{**}, \varphi^{**})\) specified below.

The equilibrium profile \((\sigma^{**}, \varphi^{**})\), where \( \sigma^{**} = (\pi^{**}, \tau^{**}) \), specifies that both types participate in the scheme \( \xi^{**} \) with probability one and the competent firm exerts high effort consistently upon participation:

\[
\pi^{**}(\xi^{**}|C) = \pi^{**}(\xi^{**}|I) = 1,
\]  

(E.47)

\[
\tau^{**}(h_2, C) = \begin{cases} 
1, & \text{if } d = \xi^{**}, \\
0, & \text{if } d = N,
\end{cases}
\]  

(E.48)

for all histories \( h_2 \in H_2 \). Consumers believe that the firm is inept for sure upon seeing a null rating, that is, \( \varphi^{**}_t(\emptyset) = 0 \) for each period \( t \).

**Proposition 8** (Singleton Optimal Rating Menu). Under Assumption 4, the menu \( \Xi^{**} \) is optimal with a rater-preferred equilibrium \((\sigma^{**}, \varphi^{**})\).

The rating 0 in period 0 does not reflect any past quality, thus \( \varphi^{**}_0(0) = \mu \). In each period \( t \geq 1 \), the two ratings statistically reveal the quality in the last period, inducing two possible levels of reputations:

\[
\varphi^{**}_t(0) = \frac{\mu(1 - \alpha(1 - \rho))}{\mu(1 - \alpha(1 - \rho)) + (1 - \mu)(1 - \alpha\rho)} < \varphi^{**}_t(1) = \frac{\mu(1 - \rho)}{\mu(1 - \rho) + (1 - \mu)\rho}.
\]

In the equilibrium \((\sigma^{**}, \varphi^{**})\), consistent high effort by the firm upon participation again implies that \( p^{**}_t(r) = \rho + (1 - 2\rho)\varphi^{**}_t(r) \). As a result, \( p^{**}_t(1) > p^{**}_t(0) \) for each \( t \geq 1 \).
Assumption 4 ensures that the competent firm exerts high effort in every period in the chase of the higher payment associated with the rating 1 in the next period.

Both ratings 0 and 1 are opaque: they do not reveal the firm’s type. Moreover, the scheme $\xi^{**}$ also dampens reputation dissipation. The worst reputation a firm can acquire by participating in the menu is $\varphi^{**}_1(0)$, which reflects either a bad quality or with some probability a good quality in the past period. The reputation dynamics is therefore suppressed, standing in stark contrast to that in $(\sigma^*, \varphi^*)$ in the game induced by $\Xi^*$. The contrast is depicted in Figure 9 below.

![Figure 9: Amplified and Dampened Reputation Dynamics](image)

The figure depicts all possible reputation paths of the competent firm in each of the rater-preferred equilibria in the induced games of interest. Specifically, it illustrates how the menu $\Xi^{**}$ amplifies both the reputation building and dissipation processes, and how the menu $\Xi^{*}$ dampens the processes on the contrary. Note that the reputation dynamics on the right panel applies also to the inept type. On the left panel, however, an inept firm’s reputation stays at the lower bound at all times except period 0.

The probability $\alpha$ is chosen so that the competent firm’s incentive constraint for high effort binds after every history upon participating in the scheme $\xi^{**}$. In particular, by creating the possibility of reaching a bad rating 0 even after a good outcome, the rating system $S^{**}$ further pools the two reputation levels closer together in the equilibrium. The competent firm is therefore indifferent between exerting each effort level after every such history, because the potential benefit of acquiring a good rating 1 balances the cost incurred after each history upon participation. An implication is that the competent type extracts the same amount of profits from consumers as an inept type does, so that the rating fee $f^{**}$ binds the participation
constraints of both types. Specifically, \( U(\sigma^{**}, C; \xi^{**}) = U(\sigma^{**}, I; \xi^{**}) = \rho + \mu(1 - 2\rho - c) \). Effectively, the rater “transfers” all the rents from the competent type to the inept type, without disrupting the competent type’s incentive for high effort.

It is then not surprising that Assumption 4 is stronger than Assumption 1 because it needs to ensure reputation effect at the prior is strong enough for high effort in a setting that largely dampens reputation building. Finally, note that optimality of the singleton menu does not rely upon the assumption that consumers do not observe the choice of the rating scheme by a firm, while the optimality of the menu \( \Xi^* \) does. If the firm’s choice of scheme is revealed to consumers in the game induced by \( \Xi^* \), the profile \((\sigma^*, \varphi^*)\) will no longer be an equilibrium, for the competent type does not consistently exert high effort once its type is revealed perfectly to consumers.

F Proofs

F.1 Proof of Proposition 8

By Proposition 1, the rater’s optimal payoff is \( \mu(1 - 2\rho - c) \) under Assumption 1. Because Assumption 4 implies Assumption 1, to prove the proposition it suffices to show Lemma 6 below.

**Lemma 6.** Under Assumption 4, \((\sigma^{**}, \varphi^{**}) \in B_2(\Xi^{**})\), and

\[
W(\Xi^*, (\sigma^{**}, \varphi^{**})) = \mu(1 - 2\rho - c).
\]

**Proof.** The proof is divided into two steps. Step 1 first shows that \((\sigma^{**}, \varphi^{**}) \in B_2(\xi^{**})\). Step 2 derives the value of \(W(\Xi^{**}, (\sigma^{**}, \varphi^{**}))\).

**Step 1.** To begin, fix the candidate equilibrium profile \((\sigma^{**}, \varphi^{**})\). Consider first the continuation game after a firm chooses participation, \(d = \xi^{**}\). Because the rating system \(S^{**}\) specified by the scheme \(\xi^{**}\) relies only on the most recent quality \(y \in Y\), and there only two possible ratings \(r \in \{0, 1\}\), and in \((\sigma, \varphi)\), the firm’s continuation effort strategy after each history is identical conditional on each rating, there are only two possible stage payoffs upon participation by a firm for each period \(t \geq 1\):

\[
p_t^{\sigma^{**}}(0) - c = \rho + (1 - 2\rho)\varphi_t^{\varphi^{**}}(0) - c = \rho + \frac{(1 - 2\rho)\mu(1 - (1 - \rho)\alpha)}{\mu(1 - (1 - \rho)\alpha) + (1 - \mu)(1 - \rho\alpha)} - c,
\]

\[
p_t^{\sigma^{**}}(1) - c = \rho + (1 - 2\rho)\varphi_t^{\varphi^{**}}(1) - c = \rho + \frac{(1 - 2\rho)\mu(1 - \rho)\alpha}{\mu(1 - \rho)\alpha + (1 - \mu)\rho\alpha} - c.
\]
Because $c \in (0, \bar{c}]$, $\alpha \in (0, 1]$ is a well-defined probability. Analogous to Lemma 2, the play from $t \geq 1$ upon participation $d = \xi^{**}$ can be represented by a two-state automaton, with the set of states given by $R = \{0, 1\}$. Here, a state $r$ collects all firm’s histories with the most recent realized rating being $r$. Let $V_C^{\sigma^{**}}(r; \xi^{**})$ be the continuation value of the competent firm in a state $r$ of the automaton given the profile $(\sigma^{**}, \varphi^{**})$ conditional on participating in $\xi^{**}$. In state $r$, by exerting high effort, the competent firm’s continuation profit is

$$V_C^{\sigma^{**}}(r; \xi^{**}) = (1 - \delta)(p_t^{\sigma^{**}}(r) - c) + \delta[(1 - \rho)\alpha V_C^{\sigma^{**}}(1; \xi^{**}) + (1 - (1 - \rho)\alpha)V_C^{\sigma^{**}}(0; \xi^{**})].$$

A deviation to shirk yields

$$V_C^{\sigma^{**}}(r; \varepsilon; \xi^{**}) := (1 - \delta)p_t^{\sigma^{**}}(r) + \delta[\rho\alpha V_C^{\sigma^{**}}(1; \xi^{**}) + (1 - \rho\alpha)V_C^{\sigma^{**}}(0; \xi^{**})].$$

For $\pi^{**}$ to be an equilibrium effort strategy conditional on $d = \xi^{**}$, the incentive constraint $V_C^{\sigma^{**}}(r; \xi^{**}) - V_C^{\sigma^{**}}(r; \varepsilon; \xi^{**}) \geq 0$ in a state $r$ needs to hold, for each $r \in \{0, 1\}$. The constraint can be simplified as

$$\delta \alpha(1 - 2\rho)^2 \left(\frac{\mu(1 - \rho)}{\mu(1 - \rho) + (1 - \mu)\rho} - \frac{\mu(1 - \alpha(1 - \rho))}{\mu(1 - \alpha(1 - \rho)) + (1 - \mu)(1 - \alpha\rho)}\right) \geq c.$$

Some algebraic manipulation reveals that, with $\alpha$ defined by (E.45), the left hand side equals $c$.\footnote{Substituting $\alpha$ defined by (E.45) on the left hand side of the inequality gives

$$\frac{c\delta(1 - 2\rho)^2(\rho + \mu(1 - 2\rho))}{c(\rho + \mu(1 - 2\rho))^2 + \delta\mu(1 - \mu)(1 - 2\rho)^3} \left(\frac{\mu(1 - \rho)}{\mu(1 - \rho) + (1 - \mu)\rho} - \frac{\delta\mu(1 - 2\rho)^2 - c\rho - c\mu(1 - 2\rho)}{\delta(1 - 2\rho)^2}\right) = c,$$

as desired.}

Therefore, the incentive constraint holds. In period 0, each type of firm receives a payment of $\rho + (1 - 2\rho)\mu$ upon participation, and the competent firm faces the same incentive constraint, which therefore is also satisfied.

Suppose now $d = N$. Consumers believe that the firm is inept for sure, and therefore pays the firm $\rho$ every period. The competent firm finds it optimal to exert low effort in every period in the continuation game.

By construction, the rating fee $f^{**}$ given by (E.46) satisfies the inept firm’s participation constraint. Observe also that upon participation, the incentive constraints for high effort bind after each rating $r$. The competent firm is therefore indifferent between high or low effort after each history upon participation, and $U(\sigma^{**}, C; \xi^{**}) = U(\sigma^{**}, I; \xi^{**})$. This implies that the participation constraint for the competent firm also holds. Thus $\pi^{**}$ is an equilibrium participation strategy, and therefore $(\sigma^{**}, \varphi^{**}) \in B_2(\Xi^{**})$.\footnote{Substituting $\alpha$ defined by (E.45) on the left hand side of the inequality gives

$$\frac{c\delta(1 - 2\rho)^2(\rho + \mu(1 - 2\rho))}{c(\rho + \mu(1 - 2\rho))^2 + \delta\mu(1 - \mu)(1 - 2\rho)^3} \left(\frac{\mu(1 - \rho)}{\mu(1 - \rho) + (1 - \mu)\rho} - \frac{\delta\mu(1 - 2\rho)^2 - c\rho - c\mu(1 - 2\rho)}{\delta(1 - 2\rho)^2}\right) = c,$$

as desired.}
Step 2. The rater’s payoff given the menu $\Xi^{**}$ in $(\sigma^{**}, \varphi^{**}) \in B_2(\Xi^{**})$ is

$$W(\Xi^{**}, (\sigma^{**}, \varphi^{**})) = (1 - \delta)[\mu f^{**} + (1 - \mu) f^{**}]$$

$$= \mu(1 - 2\rho - c),$$

as desired.

\[ \blacksquare \]

F.2 Proof of Proposition 7

The proof proceeds via a succession of lemmas. Lemma 7 establishes that $\mu(1 - c)$ is an upper bound on the rater’s payoff in an equilibrium in which each type participates in some scheme with positive probability and that the bound is tight. Lemma 8 establishes that $\mu(1 - c)$ is an upper bound on the rater’s payoff in an equilibrium in which only one type participates with positive probability. Lemma 9 then shows that this upper bound is also tight. Together, these lemmas establish that the upper bound on the rater’s payoff under perfect monitoring is $\mu(1 - c)$. To complete the proof, I finally argue that the rater achieves the upper bound if and only if either set of the stated conditions in the proposition hold.

Notice first that an identical proof of Lemma 4 holds with $\rho = 0$, so that $W_2 \leq \mu(1 - c)$. In addition, given the menu $\Xi^{*}$, the profile $(\sigma^{*}, \varphi^{*})$ remains an equilibrium in the induced game, and $W(\Xi^{*}, (\sigma^{*}, \varphi^{*})) = \mu(1 - c)$. Because the profile features participation by both types with positive probability, it follows that:

**Lemma 7.** Under Assumption 3, $W_2 = \mu(1 - c)$.

Next, consider equilibria in which only one type participates with positive probability:

**Lemma 8.** It holds that $W_1 \leq \mu(1 - c)$.

**Proof.** Fix a menu $\Xi$ and a candidate equilibrium $(\sigma, \varphi) \in B(\Xi)$ with participation strategies $\pi(\Xi|I) > 0$ and $\pi(\Xi|C) = 0$. In this candidate equilibrium, consumers believe that the participating firm is inept for sure, so that $\varphi_t(r) = 0$ and $p_{\sigma t}(r) = 0$ for any rating $r \in \bigcup_{\theta} \supp S_{\theta}(H^1_t)$ and any period $t$. The lowest reservation payoff by the inept firm to not participate is $\rho$. Respecting the inept type’s participation constraint $U(\sigma, I; \xi_\theta) - (1 - \delta)f_\theta \geq U(\sigma, I; N)$, and the fact that $U(\sigma, I; N) \geq 0$, the rater’s expected payoff satisfies

$$(1 - \delta)(1 - \mu) \sum_\theta \pi(\xi_\theta|I)f_\theta \leq (1 - \mu) \sum_\theta U(\sigma, I; \xi_\theta) = 0.$$
satisfy $\varphi_t(r) = 1$ for each rating $r \in \bigcup_{\theta \in \Theta} \text{supp} S_\theta(H^t_1)$ and each period $t$. Upon participating in a scheme $\xi_\theta$,

$$U(\sigma, C; \xi_\theta) \leq 1 - c. \quad (F.49)$$

To see this, observe first that in a candidate equilibrium where only the competent type participates with positive probability, the competent firm extracts the maximal amount from its consumers when consumers expect the firm to consistently exert high effort, and pays the firm $1 - c$ each period. This establishes (F.49). Finally, and trivially, in any equilibrium in which neither type participates, the rater’s payoff is zero. As a result, because the firm happens to be competent with probability $\mu$, and because the rater’s payoff strictly increases in the competent firm’s participation probability, $W_1 \leq \mu(1 - c)$. 

I now show that $W_1 = \mu(1 - c)$. To proceed, I construct a menu $\Xi^p = \{\xi^p\}$, where the scheme $\xi^p = (f^p, R^p, S^p)$ specifies that

$$f^p = \frac{1 - c}{1 - \delta},$$

$$R^p = \{0, 1\},$$

$$S^p(h^t_1) = \begin{cases} 1 \circ \{1\}, & \text{if there does not exist a } s < t \text{ such that } y_s = y; \\ 1 \circ \{0\}, & \text{otherwise.} \end{cases}$$

In words, so long as the participating firm does not deliver a single bad quality in the past, the rating system announces a rating 1. Consider an equilibrium $(\sigma^p, \varphi^p) \in B(\Xi^p)$, where $\sigma^p = (\pi^p, \tau^p)$, in which the competent firm participates in $\xi^p$ with probability one, the inept firm chooses the outside option with probability one, and the competent firm consistently exerts high effort upon participation and receiving a rating 1:

$$\pi^p(\xi^p|C) = 1, \quad \pi^p(\xi^p|I) = 0,$$

$$\tau^p(h^t_2, C) = \begin{cases} 1, & \text{if } d = \xi^p \text{ and } r_t = 1; \\ 0, & \text{if } d = N. \end{cases}$$

Lemma 9. $(\sigma^p, \varphi^p) \in B(\Xi^p)$ and $W(\Xi^p, (\sigma^p, \varphi^p)) = \mu(1 - c)$.

Proof. To see that $(\sigma^p, \varphi^p) \in B(\Xi^p)$, note that after any history of play from period $t \geq 0$ upon the competent firm’s participation in $(\sigma^p, \varphi^p)$, given each rating $r \in \{0, 1\}$, the firm’s stage profit and its continuation strategy by the firm are identical. Consequently the play upon participation by the competent firm can be represented by a two-state automaton,
with states given by the ratings \( r \in \{0, 1\} \) and transitions depending on the current qualities, depicted by Figure 10 below. Each state in the automaton represents an equivalence class of a firm’s history, where each member of an equivalence class induces an identical expected flow profit and an identical continuation strategy. Here, a state \( r \) collects all firm’s histories with the most recent realized rating being \( r \). The initial state is 1, in which the competent firm chooses \( \bar{e} \) with probability 1; in state 0, the competent firm chooses \( \bar{e} \) with probability 1.

![Figure 10: The two-state automation](image)

It can readily be verified that \((\sigma^p, \varphi^p)\) is an equilibrium in the continuation game upon participation in the scheme \( \xi^p \) using the automaton. Denoting the competent firm’s continuation profit extraction from consumers in a state \( r \in \{0, 1\} \) by \( V_{\sigma^p}^r \), it holds that

\[
V_{\sigma^p}^1 = (1 - \delta)(1 - c) + \delta V_{\sigma^p}^1 \\
V_{\sigma^p}^0 = 0.
\]

Solving the system yields \( V_{\sigma^p}^1 = 1 - c \). First, the incentive constraint for the specified effort choices in state 1, namely \( \delta(V_{\sigma^p}^1 - V_{\sigma^p}^0) \geq (1 - \delta)c \), is satisfied, because it is equivalent to Assumption 3. Also, clearly, the competent firm finds it optimal to exert low effort in state 0.

It remains to verify that in \((\sigma^p, \varphi^p)\), both types do not find it profitable to deviate from the specified participation choice. By deviating to not participate in \( \xi^p \), the competent firm finds it optimal to always exert low effort, because future consumers do not see the outcome it produces in each period. Each entering consumer thus pays the firm exactly 0 upon observing a null rating, giving \( U(\sigma^p, C; N) = 0 \). Given the rating fee \( f^p \) in the scheme, in \((\sigma^p, \varphi^p)\), the competent firm obtains

\[
U(\sigma^p, C; \xi^p) - (1 - \delta)f^p = 1 - c - (1 - c) = 0 = U(\sigma^p, C; N).
\]

Similarly, by not participating, the inept firm obtains \( U(\sigma^p, I; N) = 0 \). If it deviates to participate in \( \xi^p \), then it obtains

\[
U(\sigma^p, I; \xi^p) - (1 - \delta)f^p = (1 - \delta)1 - (1 - c) \leq 0 = U(\sigma^p, I; N).
\]
where the inequality follows from Assumption 3. Therefore \((\sigma^p, \varphi^p) \in B(\Xi^p)\), and the rater obtains

\[
W(\Xi^p, (\sigma^p, \varphi^p)) = \mu(1 - \delta)f^p = \mu(1 - c).
\]

It remains to argue that the rater achieves the payoff upper bound if and only if the stated conditions in the proposition hold. Consider first equilibria in which each type participates in some scheme with positive probability. It follows analogously from Proposition 1 that the rater achieves the upper bound if and only if the first set of the conditions hold. Next, observe that in any equilibrium in which only one type participates with positive probability, the rater obtains a payoff \(\mu(1 - c)\) if and only if first, the competent type participates with probability one; second, it consistently exerts high effort upon participation, and the rating fees in the schemes in which the competent type participates with positive probability fully extract all its profits above the minimal compensation \(\rho\). This completes the proof.

\section{G Omitted Calculation: Competent Firm’s Maximal Profit}

This section collects some detailed calculations omitted from the main text. I derive the competent firm’s profit upper bound (16) in the canonical setting with both adverse selection and moral hazard. Let \(V^\sigma(h_2)\) denote the competent firm’s continuation profit in some equilibrium \((\sigma, \varphi)\) after a history \(h_2\), and let \(V^\sigma(y; h_2)\) denote the expected continuation profit upon a quality realization \(y\) in the game after history \(h_2\). The incentive constraint for high effort after each history \(h_2\) is

\[
V^\sigma(\bar{y}; h_2) - V^\sigma(y; h_2) \geq \frac{(1 - \delta)c}{\delta(1 - 2\rho)}.
\]

Because high effort is efficient and respecting the incentive constraint (G.50), the competent firm’s equilibrium expected profit is

\[
V^\sigma(h_2^0) \leq (1 - \delta)(\rho + \mu(1 - 2\rho) - c) + \delta((1 - \rho)V^\sigma(\bar{y}; h_2^0) + \rho V^\sigma(y; h_2^0))
\]

\[
\leq (1 - \delta)(\rho + \mu(1 - 2\rho) - c) + \delta \left( V^\sigma(\bar{y}; h_2^0) - \frac{(1 - \delta)c}{\delta(1 - 2\rho)} \right)
\]
\[
\leq (1 - \delta)(\rho + \mu(1 - 2\rho) - c) + \delta \left((1 - \delta)(\rho + \frac{\mu(1 - \rho)(1 - 2\rho)}{\mu(1 - \rho) + (1 - \mu)\rho}) - c \right.
\]
\[
+ \delta((1 - \rho)V^\sigma(\bar{y}; h_1^2) + \rho V^\sigma(\bar{y}; h_2^1)) - \frac{(1 - \delta)\rho c}{\delta(1 - 2\rho)})
\]
\[
\leq \cdots
\]
\[
\leq (1 - \delta)\left(\sum_{t=0}^{\infty} \delta^t \rho + \sum_{t=0}^{\infty} \delta^t \frac{(1 - 2\rho)(1 - \rho)^t \mu}{\mu(1 - \rho)^t + (1 - \mu)\rho^t} - \sum_{t=0}^{\infty} \delta^t c - \sum_{t=0}^{\infty} \delta^t \frac{\rho c}{1 - 2\rho} \right)
\]
\[
= \rho + (1 - \delta)(1 - 2\rho) \sum_{t=0}^{\infty} \delta^t \frac{(1 - \rho)^t \mu}{\mu(1 - \rho)^t + (1 - \mu)\rho^t} - c - \frac{\rho c}{1 - 2\rho},
\]
as desired.